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ON DUALITIES FOR STRONGLY DECOMPOSABLE OPERATORS

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1. Notations and definitions

Throughout this note, X denotes a complex Banach space, $B(X)$ the Banach algebra of all bounded linear operators of X and X^* the dual of X. For an operator $T\in B(X)$, T^* denotes the dual operator of T. If M is a closed T-invariant subspace of X, we write $T|M$ for the restriction and T/M for the operator induced by T on the quotient space X/M. For $N\subset X$, let $N^{\perp}$ be its annihilator in X^*, $\overline{N}$ the closure of N. The symbol $\sigma(T)$ stands for the spectrum of T. We denote $\mathcal{U}$ and $\mathcal{F}$ the class of all open subsets and the closed subsets in the finite complex plane $\boldsymbol{C}$ respectively. If T has the single valued extension property, we denote $X_T(F)=\{x\in X : \sigma(x, T)\subset F\}$. This is a linear subspace of X but not necessarily closed even if F is closed in $\boldsymbol{C}$. The set theoretic difference between two sets A and B is denoted by $A-B$.

DEFINITION 1.1 ([3]). Let $T\in B(X)$. A T-invariant subspace Z is said to be spectral maximal for T if for any T-invariant subspace Y such that $\sigma(T|Y)\subset\sigma(T|Z)$ we have that $Y\subset Z$.

We denote the set of all spectral maximal subspaces for T by SM(T).

DEFINITION 1.2 ([3]). An operator $T\in B(X)$ is said to be decomposable if for any finite system $\{G_1, G_2, \ldots\ldots, G_n\}$ of open subsets of $\boldsymbol{C}$ that cover $\sigma(T)$, there exist spectral maximal subspaces $\{Y_1, Y_2, \ldots\ldots, Y_n\}$ such that $X=\sum_{i=1}^{n} Y_i$ and $\sigma(T|Y_i)\subset G_i$ for $i=1, 2, \ldots\ldots, n$.

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It is known that if T is decomposable, then $\mathrm{SM}(T)=\{X_T(F): F\in\mathcal{F}\}$.

DEFINITION 1.3 ([9]). The T-invariant subspace Y is called analytically invariant if for each X-valued analytic function f defined on a region V_f in $\boldsymbol{C}$ such that $(\lambda-T)f(\lambda)\in Y$ for $\lambda\in V_f$, then it follows that $f(\lambda)\in Y$ for $\lambda\in V_f$.

It is known that "Spectral Maximal" implies "Analytically invariant" but the converse is false. We denote the class of all analytically invariant subspaces for T by $\mathrm{AI}(T)$. Thus $\mathrm{SM}(T)\subset\mathrm{AI}(T)$.

DEFINITION 1.4 ([6]). A decomposable operator is strongly decomposable if the operator $T|Y$ is decomposable for every T-spectral maximal subspace Y.

2. Analytical spectral resolvent (ASR)

DEFINITION 2.1. A map $E:\mathcal{U}\rightarrow\mathrm{AI}(T)$ is said to be an analytic spectral resolvent of T if

(i) $E(\phi)=\{0\}$,

(ii) For any finite open cover $\{G_1, G_2, \ldots\ldots, G_n\}$ of $\sigma(T)$,

$$X=\sum_{i=1}^{n}E(G_i),$$

(iii) $\sigma(T|E(G))\subset\overline{G}$ for each $G\in\mathcal{U}$.

Thus an ASR is a spectral resolvent, which is defined in [5], whose range is analytically invariant subspaces.

The ASR for T is not unique as well as the spectral resolvent, there are typical types of ASR for T.

REMARK 2.2. Let T be strongly decomposable, then the map E defined by $E(G)=\overline{X_T(G)}$ $(G\in\mathcal{U})$ is an ASR for T.

For, it is known that if T is decomposable then $\overline{X_T(G)}$ is analytically invariant for each $G\in\mathcal{U}$. Obviously $E(\phi)=\overline{X_T(\phi)}=\{0\}$, and $\sigma(T|E(G))=\sigma(T|\overline{X_T(G)})\subset\sigma(T|X_T(\overline{G}))\subset\overline{G}(G\in\mathcal{U})$ hold since $\overline{X_T(G)}\subset X_T(\overline{G})$, and both $\overline{X_T(G)}$, $X_T(\overline{G})$ are analytically invariant under T; in fact, $X_T(\overline{G})$ is spectral maximal so it is analytically invariant. For any finite open cover $\{G_1, G_2, \ldots\ldots, G_n\}$ of $\sigma(T)$,

$$\sum_{i=1}^{n} E(G_i) = \sum_{i=1}^{n} \overline{X_T(G_i)} \supset \sum_{i=1}^{n} X_T(G_i) = X_T(\bigcup_{i=1}^{n} G_i) = X_T(\sigma(T)) = X,$$

the second equality holds since T is strongly decomposable (see [6], p. 86, Lemma 12. 7).

REMARK 2. 3. Let T be decomposable. The map E defined by $E(G) = X_T(\overline{G})$ $(G \in \mathcal{U})$ is an ASR for T.

Proof. Let $\{G_i\}_{i=1}^{n}$ be any open covering of $\sigma(T)$, it is known that

$$X = X_T(\sigma(T)) \subset \sum_{i=1}^{n} X_T(\overline{G}_i), \text{ thus } X = \sum_{i=1}^{n} X_T(\overline{G}_i).$$

Obviously, $\sigma(T | X_T(\overline{G})) \subset \overline{G}$ and $X_T(\{0\}) = \phi$.

THEOREM 2. 4. *If T has an ASR $E : \mathcal{U} \to AI(T)$, then T is decomposable.*

There are three different methods of proof on this theorem. Among those we give a proof using the following theorem.

THEOREM 2. 5 ([10]). *For an operator T, the following are equivalent.*

(a) *T is decomposable.*

(b) *For every open set G in $\mathbf{C}$, there is a T-invariant subspace M such that $\sigma(T | M) \subset G$ and $\sigma(T/M) \subset \mathbf{C} - G$.*

Proof of Theorem 2. 4. Since $\sigma(T | E(G)) \subset \overline{G}$ by definition, and $\sigma(T/E(G)) \subset \mathbf{C} - G$ holds if $E(G)$ is analytically invariant under T (see [5], p. 60, Theorem 10). (In fact, $\sigma(T/E(G) \subset \sigma(T) - G$ since $\sigma(T/E(G)) \subset \sigma(T))$. Hence the conclusion follows by Theorem 2. 5.

Further properties for an operator T having ASR were studied in [11].

3. A duality theorem for a strongly decomposable operator

In this section, we prove the main result, that is, if T is strongly decomposable with the spectrum $\sigma(T)$ of T, under some conditions, the dual operator T^* of T is strongly decomposable.

To begin with we list here some basic results.

PROPOSITION 3.1 ([1], p. 1; [9], p. 231). *Let Y and Z be T-invariant subspaces such that $Y \subset Z$. Then*

(1) $Y \in AI(T)$ *implies* $Y \in AI(T|Z)$

(2) $Y \in AI(T|Z)$, $Z \in AI(T)$ *implies* $Y \in AI(T)$

(3) $Z \in AI(T)$ *if and only if* $Z/Y \in AI(T/Y)$

(4) $(T|Z)|Y = T|Y$

(5) $(T|Z)/Y = (T/Y)|(Z/Y)$

We prove the following lemma using the above proposition.

LEMMA 3.2. *Let T be decomposable. For an open set G in $\boldsymbol{C}$, we put $Y(G) = \overline{X_T(G)}$, $Z(G) = X_T(\overline{G})$, $\tilde{Y}(G) = Z(G)/Y(G)$, $\tilde{T} = T/Y(G)$ $\hat{X}(G) = X/Y(G)$ and $\tilde{T}^*$ is the dual operator of $\tilde{T}$. Then*

$$\sigma(\tilde{T}|\tilde{Y}(G)) \subset \overline{G}, \quad \sigma(\tilde{T}/\tilde{Y}) \subset \boldsymbol{C} - G$$

and $\tilde{Y}$ is analytically invariant under $\tilde{T}$.

Proof. Let G be arbitrary open in $\boldsymbol{C}$ but fixed, let $Y = Y(G)$, $Z = Z(G)$ and $\tilde{Y} = \tilde{Y}(G)$. By proposition 3.1, (3), $\tilde{Y} = Z/Y$ is analytically invariant under $\tilde{T} = T/Y$ since both Y and Z are analytically invariant under T. Since T is decomposable, $Y = \overline{X_T(G)}$ is analytically invariant under T, it is also analytically invariant under $T|Z$. Thus we have

$$\sigma[(T|Z)/Y] \subset \sigma(T|Z) = \sigma(T|X_T(\overline{G})) \subset \overline{G};$$

the first inclusion follows from the fact that, in general, if Y is analytically invariant (or spectral maximal) under T, then $\sigma(T) = \sigma(T|Y) \cup \sigma(T/Y)$ (see [9], p. 227, Proposition 1.5).

Moreover, from the equality $(T|Z)/Y = (T/Y)|(Z/Y)$, we have

$$\sigma(\tilde{T}|\tilde{Y}) = \sigma[(T/Y)|(Z/Y)] = \sigma[(T|Z)/Y] \subset \overline{G}.$$

Since $G \in \mathfrak{U}$ is arbitrary, we have $\sigma(\tilde{T}|\tilde{Y}(G)) \subset \overline{G}$ for any $G \in \mathfrak{U}$.

Again fix G. By the identification $(T/Z)^* = T^*|Z^{\perp} = T^*|X_T(G)^{\perp}$, we get

$$\sigma(T^*|X_T(\overline{G})^{\perp}) = \sigma[(T/Z)^*] = \sigma(T/Z).$$

Furthermore, since $Y \subset Z$, we have the following unitarily equivalence relation

$(T/Y)^*|(Z/Y)^{\perp} \cong T^*|Z^{\perp}$ (see [7], p. 292, Lemma 5).

Therefore

$$\sigma((\tilde{T})^*|\tilde{Y}^{\perp}) = \sigma(T^*|Z^{\perp}) = \sigma[(T/Z)^*] = \sigma(T/Z) = \sigma(T/X_T(\overline{G})) \subset \boldsymbol{C} - G,$$

the last inclusion holds since $Z(G)=X_T(\overline{G})$ $(=E(G))$ defines an ASR for T as we noted in Remark 2.3. In fact $\sigma(T/X_T(\overline{G}))\subset\sigma(T)-G$ since $\sigma(T/X_T(\overline{G}))\subset\sigma(T)$.

The arbitrariness of G implies $\sigma[(\tilde{T})^*|\tilde{Y}(G)^{\perp}]\subset \boldsymbol{C}-G$ for every $G\in\mathfrak{U}$. It follows that $\sigma(\tilde{T}/\tilde{Y})=\sigma[(\tilde{T}/\tilde{Y})^*]=\sigma[(\tilde{T})^*|\tilde{Y}(G)^{\perp}]\subset\boldsymbol{C}-G$. We have proved the lemma.

Now, we consider again the identification $(T/\overline{X_T(G)})^*=T^*|X_T(G)^{\perp}$. Since $\mathrm{SM}(T^*)=\{X_T(\boldsymbol{C}-F)^{\perp}: F\in\mathfrak{F}\}=\{X_T(G)^{\perp}: G\in\mathfrak{U}\}$ (see [8], p. 1057, Remark), T^* is strongly decomposable if and only if $T^*|X_T(G)^{\perp}$ is decomposable for every $G\in\mathfrak{U}$. Therefore, T^* is strongly decomposable if and only if $T/\overline{X_T(G)}$ is decomposable for every $G\in\mathfrak{U}$ since, in general, $A\in B(X)$ is decomposable if and only if A^* is.

It is known that if T is strongly decomposable then T/M is decomposable for any spectral maximal space M for T. Since $X_T(G)=X_T(G\cap\sigma(T))$, if $\sigma(T)$ is finite then $T/X_T(G)$ is decomposable for any $G\in\mathfrak{U}$, whence T^* is strongly decomposable.

Thus we have the following

Proposition 3.3. *Let T be strongly decomposable. If the spectrum of T is finite, then T^* is strongly decomposable.*

Theorem 3.4. *Let T be strongly decomposable. If the spectrum $\sigma(T)$ of T does not contain any isolated point, the interior of $\sigma(T)=G_\circ$ is nonempty and $\overline{X_T(G_\circ)}=X$ then T^* is strongly decomposable.*

Proof. For those open sets such that $G\cap\sigma(G)=\phi$, $X_T[G\cap\sigma(T)]=\{0\}$, whence $T/\overline{X_T(G)}=T$ is decomposable. So we may assume without loss of generality that $G\cap\sigma(T)\neq\phi$. Let $G\in\mathfrak{U}$ be arbitrary but fixed, and let H be any open set in $\boldsymbol{C}$. We put $Y=\overline{X_T(G)}$, $Z=\overline{X_T(G\cup H)}$, $\tilde{Y}=Z/Y$, $\tilde{T}=T/Y$ and let $(\tilde{T})^*$ be the dual of $\tilde{T}$.

By the Similar proof as that of Lemma 3.2, $\tilde{Y}$ is analytic invariant under $\tilde{T}$. Now, we prove that

(*) $\sigma(\tilde{T}|\tilde{Y})\subset\overline{H}$, $\sigma(\tilde{T}/\tilde{Y})\subset\boldsymbol{C}$-$H$ for any $H\in\mathfrak{U}$.

Then, by Theorem 2.5., $\tilde{T}=T/X_T(G)$ is decomposable. Arbitrariness of G implies that T^* is strongly decomposable.

For an open set H such that $\sigma(\tilde{T})\cap H=\phi$,

$$\sigma(\tilde{T}\,|\,\{0\})=\phi\subset H,\quad \sigma(\tilde{T}/\{0\})=\sigma(\tilde{T})\subset \boldsymbol{C}-H;$$

where 0 is the zero vector in $X/\overline{X_T(G)}$, that is, $\overline{X_T(G)}=0$. Therefore, without loss of generality, we may assume that $\sigma(\tilde{T})\cap H\neq\phi$. Since $\sigma(\tilde{T})=\sigma(T/\overline{X_T(G)})\subset\boldsymbol{C}-G$, $\sigma(\tilde{T})\cap H\neq\phi$, so $H-G\neq\phi$ and $\sigma(T)\cap H\neq\phi$.

Case (a). $\sigma(T|X_T(G\cup H))\neq\sigma(T)$.

Since $T|Z=T|\overline{X_T(G\cup H)}$ is decomposable, we have

(**) $\sigma[T|\overline{X_T(G\cup H)}]-\overline{G}\subset\sigma(T|Z)-\sigma(T|Y)$

$$=\sigma(T|Z)-\sigma[(T|Z)\,|\,Y]\subset\sigma[(T|Z)/Y]=\sigma(\tilde{T}|\,\tilde{Y})\subset\sigma(T|Z)-G;$$

the last inclusion holds since $Y\in AI(T|Z)$, so $\sigma[(T|Z)/Y]\subset\sigma(T|Z)$; and since $\tilde{Y}\in AI(\tilde{T})$, $\sigma(\tilde{T}|\,\tilde{Y})\subset\sigma(\tilde{T})=\sigma(T/\overline{X_T(G)})\subset\boldsymbol{C}-G$.

As we stated in Remark 2.2, $E(G)=\overline{X_T(G)}$ defines an ASR for T, $\overline{\sigma(T)\cap G}\subset\sigma(T|\overline{X_T(G)})\subset\overline{G}\cap\sigma(T)$ $(G\in\mathfrak{U})$ (see [12], p. 81, Prop. 17), and $\overline{(G\cup H)\cap\sigma(T)}\subset\sigma(T|Z)\subset\overline{G\cup H}\cap\sigma(T)$. Moreover since $(G\cup H)\cap\sigma(T)\neq\phi$, $\sigma(T)$ contains no isolated point, so $\overline{(G\cup H)\cap\sigma(T)}=\overline{G\cup H}\cap\sigma(T)$, also $\overline{G\cap\sigma(T)}=\overline{G}\cap\sigma(T)$.

Thus we have

$$\sigma(T|Z)-\overline{G}=\sigma(T|Z)-\overline{G}\cap\sigma(T|Z)\supset\overline{(G\cup H)\cap\sigma(T)}-\overline{G}\cap\sigma(T|Z)$$
$$=[\overline{G}\cap\sigma(T)]\cup[\overline{H}\cap\sigma(T)]-\overline{G}\cap\sigma(T|Z)\neq\phi.$$

We claim that

$$[\sigma(T|Z)-\overline{G}]^-=\sigma(T|Z)-G.$$

Suppose $[\sigma(T|Z)-G]^-\subsetneqq\sigma(T|Z)-G$. Choose λ belong to the right but not the left, then dist. $\{\lambda,[\sigma(T|Z)-\overline{G}]^-\}>0$.

While $\lambda\in[\sigma(T|Z)-G]-[\sigma(T|Z)-\overline{G}]=\sigma(T|Z)\cap\partial G$, where $\partial G=\overline{G}-G$, the boundary of G. But this implies $\lambda\in[\sigma(T|Z)-\overline{G}]^-$, which is a contradiction. Therefore, we get, by (**), that

$$\sigma(\tilde{T}|\,\tilde{Y})=[\sigma(T|Z)-\sigma(T|Y)]^-\subset[\overline{G\cup H}\cap\sigma(T)-\overline{G\cap\sigma(T)}]^-$$
$$=[(\overline{G}\cap\sigma(T))\cup(\overline{H}\cap\sigma(T))-\overline{G}\cap\sigma(T)]^-\subset\overline{H}\cap\sigma(T)\subset\overline{H}.$$

i.e. $\sigma(\tilde{T}|\,\tilde{Y})\subset\overline{H}$.

Case (b). $\sigma(T|X_T(G\cup H)=\sigma(T)$.

In this case, (**) can be written by

$$\sigma(T)-\overline{G}\subset\sigma(T)-\sigma(T|Y)\subset\sigma(T/Y)\subset\sigma(T)-G.$$

(i) If $\sigma(T)-G=\phi$ then $\sigma(\tilde{T}|\tilde{Y})\subset\sigma(\tilde{T})=\sigma(T/Y)=\phi$.
Thus $\sigma(\tilde{T}|\tilde{Y})\subset\overline{H}$ for any $H\in\mathfrak{U}$.

(ii) If $\sigma(T)-\overline{G}\neq\phi$, then, by the same calculation as in (a), we have

$$[\sigma(T)-\overline{G}]^{-}=\sigma(T)-G, \text{ so } [\sigma(T)-\sigma(T|Y)]^{-}=\sigma(T/Y).$$

Hence $\sigma(\tilde{T}|\tilde{Y})\subset\sigma(T/Y)=[\sigma(T)-\sigma(T|Y)]^{-}$
$$\subset[\overline{G}\cup\overline{H})\cap\sigma(T)-\overline{G}\cap\sigma(T)]^{-}\subset\overline{H}.$$

(iii) Finaly, if $\sigma(T)-\overline{G}=\phi$ but $\sigma(T)-G\neq\phi$, then
$$X=\overline{X_T(G_o)}\subset\overline{X_T(G\cap\sigma(T))}.$$

Thus $X/\overline{X_T(G)}$ is the zero vector. Therefore, we have
$$\sigma(\tilde{T}|\tilde{Y})=\sigma[(T/Y)|X/\overline{X_T(G)})]=\phi\subset\overline{H}.$$

For the second inclusion, the proof is the same as that of Lemma 3.2; by the identification $(T/Y)^*(Z/Y)^{\perp}=T^*|Z^{\perp}$, we have
$$\sigma(\tilde{T}/\tilde{Y})=\sigma[(\tilde{T}/\tilde{Y})^*]=\sigma[(\tilde{T})^*|\tilde{Y}^{\perp}]=\sigma(T^*|Z^{\perp})$$
$$=\sigma[(T/Z)^*]=\sigma(T/Z)\subset\boldsymbol{C}-(G\cup H)\subset\boldsymbol{C}-H.$$
We completes the proof.

Example 3.5. Let T be strongly decomposable with the spectrum $\sigma(T)=[a,b]$, $a<b$. We prove that T^* is strongly decomposable. According to the Theorem 3.4, it is enough to show that $\overline{X_T[(a,b)]}=X$ since $G_o=(a,b)$.

We choose a system of open sets $G_n=\left(a-\frac{1}{n},\ a+\frac{1}{n}\right)\cup\left(b-\frac{1}{n},\ b+\frac{1}{n}\right)$ in $\boldsymbol{R}$, $n=1,2,\ldots\ldots$ Then $\overline{G}_{n+1}\subset\overline{G}_n$ for $n=1,2,\ldots\ldots$, whence

$$X_T(\overline{G}_{n+1})\subset X_T(\overline{G}_n), \overline{G}_n\cap\sigma(T)=[a,a+\frac{1}{n}]\cup(b-\frac{1}{n},b], \text{ and}$$

$$\bigcap_{n=1}^{\infty}[\overline{G}_n\cap\sigma(T)]=\lim_{n\to\infty}[\overline{G}_n\cap\sigma(T)]=\{a,b\}=\partial_R\sigma(T).$$

In general, for any system of open sets $\{H_i\}_{i=1}^{n}$ in $\boldsymbol{C}$, $X_T(\bigcup_{i=1}^{n}H_i)=\sum_{n=1}^{n}X_T(H_i)$ holds if T is strongly decomposable (see [6], p. 86, Lemma 12.7). Therefore, we have
$$X=X_T([a,b])\subset X_T[(a,b)\cup G_n]=X_T[(a,b)]+X_T(G_n).$$
Thus

$$X\subset\overline{X_T[(a,b)]}+\bigcap_{n=1}^{\infty}X_T[\overline{G}_n\cap\sigma(T)]=\overline{X_T[(a,b)]}+X_T[\bigcap_{n=1}^{\infty}\overline{G}_n\cap\sigma(T)]$$
$$=\overline{X_T[(a,b)]}+X_T[\partial_R\sigma(T)]=\overline{X_T[(a,b)]}+X_T[\{a,b\}].$$

Moreover, since two closed sets $\{a\}$, $\{b\}$ are disjoint
$X_T[\{a,b\}]=X_T(\{a\})\oplus X_T[\{b\}]$.

Both $X_T[\{a\}]$ and $X_T[\{b\}]$ are contained in $\overline{X_T[(a,b)]}$; for, let $\{\lambda_n\}$ be a sequence in (a,b) such thabt $\lambda_n\to b$ as $n\to\infty$.
Since

$$X_T(\{\lambda_1,\lambda_2,\dots,\lambda_n\})\subset X_T(\{\lambda_1,\lambda_2,\dots,\lambda_n,\lambda_{n+1}\})\subset\overline{X_T(a,b)}$$

hold for any $n\in N$, whence

$$X_T[\{b\}]\subset X_T(\{\lambda_1,\lambda_2,\dots,\lambda_n,\dots,b\})=\lim_{n\to\infty}X_T(\{\lambda_1,\lambda_2,\dots,\lambda_n\})$$
$$\subset\overline{X_T[(a,b)]}.$$

Similary, $X_T[\{a\}]\subset\overline{X_T[(a,b)]}$. Hence

$$X_T[\{a,b\}]=X_T[\{a\}]\oplus X_T[\{b\}]\subset\overline{X_T[(a,b)]},$$

and

$$X=X_T[(a,b)].$$

THEOREM 3.6. *Let $A=C[a,b]$ be the commutative Banach *-algebra of complex-valued continuous functions on $[a,b]$ endowed with the norm $\|x\|=\sup_{t\in[a,b]}|x(t)|$ $(x\in A)$ and the natural involution. The operator T of multiplication by independent variables in $C[a,b]$ defined by $(Tx)(t)=tx(t)$ $(t\in[a,b])$ is strongly decomposable and $\sigma(T)=[a,b]$.*

Proof. Let $m\in C[a,b]$ be $m(t)=t$, $t\in[a,b]$. The multiplication operator T_m defined by $T_mx=mx$. Since

$$(T_mx)(t)=m(t)x(t)=tx(t),\ \text{so}\ T=T_m.$$

We prove that T_m is strongly decomposable: Since $[a,b]$ is compact Hausdorff for the usual topology, the maximal ideal space of $A=C[a,b]$ is $[a,b]$ (See [13], p. 271, Example (a)). For every closed subset F of $[a,b]$ and $t_0\notin F$, there exists an $x\in C[a,b]$ such that $x=0$ on F and $x(t_0)\neq 0$ thus $C[a,b]$ is regular. By the Gelfand-Naimark theorem, A is also semisimple, whence every multiplication operator in A is super-decomposable (See [11], p. 42, Corollary 2.4), so it is strongly decomposable (See [11], p. 36, Theorem 1.3).

The fact $\sigma(T)=[a,b]$ is well known.

COROLLARY 3.7. *The operator of multiplication by independent variables* *in* $A=C[a,b]$ *and its dual are strongly decomposable.*

This follows from Example 3.5 and Theorem 3.6.

For the representation of $T_m{}^*$, we consider $A=C[a,b]$ as a Banach space, let A^* be its dual. By the Riesz's representation theorem $T_m{}^*$ can be represented by Riemann-Stieltjes integral

$$(T_m{}^*f)(x)=f(T_m x)=\int_a^b tx(t)\,dw(t) \quad (x\in A,\ f\in A^*),$$

where w is a bounded variation function on $[a,b]$ and has the total variation $\mathrm{Var}(w)=\|f\|$.

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