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# REPRESENTATIONS OF CERTAIN MEDIAL ALGEBRAS

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## 1. Introduction

Let $(A, \Omega)$ be an (abstract) algebra. We say an $m$-ary operation $f$ and an $n$-ary operation $g$ in $\Omega$ *commute* if

$$f(g(x_{11}, x_{12}, \cdots, x_{1n}), g(x_{21}, x_{22}, \cdots, x_{2n}), \cdots, g(x_{m1}, x_{m2}, \cdots, x_{mn}))$$
$$=g(f(x_{11}, x_{21}, \cdots, x_{m1}), f(x_{12}, x_{22}, \cdots, x_{m2}), \cdots, f(x_{1n}, x_{2n}, \cdots, x_{mn}))$$

for all $x_{ij}$ in $A$, $i=1, 2, \cdots, m$, $j=1, 2, \cdots, n$. An algebra is called *medial* if every pair of operations (not necessarily distinct) commute.

Let $(A, f, g)$ be a medial algebra with an $m$-ary operation $f$ and an $n$-ary operation $g$. Since any unary operation of a medial algebra is nothing more than a homomorphism of the algebra, we may assume $2 \leq m \leq n$. For any element $e$ of $A$, let $\sigma_1, \cdots, \sigma_m$ and $\tau_1, \cdots, \tau_n$ be mappings of $A$ into $A$ defined by

$$\sigma_i : x \to f(e, \cdots, e, x, e, \cdots, e) \text{ and } \tau_i : x \to g(e, \cdots, e, x, e, \cdots, e) \quad (1)$$

with $x$ at the $i$-th place. We call $\sigma_i$ *the $i$-th translation by $e$ with respect to $f$*. An element $e$ is called *$i$-regular* (resp. an *$i$-identity*) with respect to $f$ if $\sigma_i$ is a bijection (resp. the identity mapping). The similar definitions go with $g$. An element $c$ is called *regular* (resp. *an identity*) if it is *$i$-regular* (resp. an $i$-identity) with respect to both $f$ and $g$ for all $i$. Finally, an element $e$ is called *idempotent* if $f(e, e, \cdots, e) = g(e, e, \cdots, e) = e$.

It is known that any medial algebra in certain varieties (varieties in which every algebra has a modular lattice of congruences) can be represented as a module over a commutative ring ([2], [4], [5]). However, the condition is very strong and we want to replace the condition with weaker condition and still obtain a representation of a medial algebra as a familiar algebra. In this paper, the condition is

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replaced with the condition that the algebra has an idempotent regular element. Quite a lot work has been done in this way for medial groupoids ([1], [3], [6]), and there are some results for medial algebras with one operation ([3]). We will see that any medial algebra with an idempotent regular element can be reconstructed from a monoid. We can extend this result slightly to medial groupoids without regular elements, but with elements which are idempotent and $i$-, $j$-regular for two different $i$ and $j$.

## 2. Medial algebras with idempotent regular elements

LEMMA 1. *Let $(A, f, g)$ be a medial algebra with an identity element $e$, then*

$$f(x_1, x_2, \cdots, x_m) = g(x_{\pi 1}, x_{\pi 2}, \cdots, x_{\pi m}, e, \cdots, e)$$

*for any permutation $\pi$ on $\{1, 2, \cdots, m\}$ and for all $x_1, x_2, \cdots, x_m$ in $A$.*

*Proof*. For any permutation $\pi$ on $\{1, 2, \cdots, m\}$ and $x_1, x_2, \cdots, x_m$ in $A$,

$$\begin{aligned} & f(x_1, x_2, \cdots, x_m) \\ &= f(g(e, \cdots, e, \underset{\pi^{-1}1\text{-th}}{x_1}, e, \cdots, e), \cdots, g(e, \cdots e, \underset{\pi^{-1}m\text{-th}}{x_m}, e, \cdots, e)) \\ &= g(f(e, \cdots, e, \underset{\pi 1\text{-th}}{x_{\pi 1}}, e, \cdots, e), \cdots, f(e, \cdots, e, \underset{\pi m\text{-th}}{x_{\pi m}}, e, \cdots, e), \cdots, g(e, \cdots, e)) \\ &= g(x_{\pi 1}, \cdots, x_{\pi m}, e, \cdots, e). \end{aligned}$$

COROLLARY. *If $(A, f)$ is a medial $n$-groupoid with an identity element, then*

$$f(x_1, x_2, \cdots, x_n) = f(x_{\pi 1} x_{\pi 2}, \cdots, x_{\pi n})$$

*for any permutation $\pi$ on $\{1, 2, \cdots, n\}$ and for all $x_1, x_2, \cdots, x_n$ in $A$.*

LEMMA 2. ([1]). *Every medial groupoid with an identity element is a commutative semigroup.*

THEOREM 3. *If $(A, f, g)$ is a medial algebra with an identity element $e$, then there is a commutative semigroup $(A, +)$ with $e$ as the identity element such that*

$f(x_1, x_2, \cdots, x_m) = x_1 + x_2 + \cdots + x_m$ and $g(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n$ *for all $x_1, \cdots, x_m, \cdots, x_n$ in $A$.*

*Proof*. Define a binary operation '+' on $A$ by

$$x+y=f(x, y, e, \cdots, e) \tag{2}$$

for all $x, y$ in $A$. We note that $x+y=g(x, y, e, \cdots, e)$ by Lemma 1. For $x, y, z, w \in A$,

$$\begin{aligned}
&(x+y)+(z+w)\\
&=f(f(x, y, e, \cdots, e), f(z, w, e, \cdots, e), e, \cdots, e)\\
&=f(f(x, y, e, \cdots, e), f(z, w, e, \cdots, e), f(e, \cdots, e), \cdots, f(e, \cdots, e))\\
&=f(f(x, z, e, \cdots, e), f(y, w, e, \cdots, e), f(e, \cdots, e), \cdots, f(e, \cdots, e))\\
&=f(f(x, z, e, \cdots, e), f(y, w, e, \cdots, e), e, \cdots, e)\\
&=(x+z)+(y+w).
\end{aligned}$$

Thus, $(A, +)$ is medial. Trivially, $e$ is the identity element of $(A, +)$, and so $(A, +)$ is a commutative semigroup by Lemma 2. Suppose $f(x_1, \cdots, x_i, e, \cdots, e)=x_1+\cdots+x_i$, then

$$\begin{aligned}
&f(x_1, \cdots, x_i, x_{i+1}, e, \cdots, e)\\
&=f(f(x_1, e, \cdots, e), \cdots, f(x_i, e, \cdots, e), f(e, x_{i+1}, e, \cdots, e), (f(e, \cdots, e),\\
&\qquad \cdots, f(e, \cdots, e))\\
&=f(f(x_1, \cdots, x_i, e, \cdots, e), f(e, \cdots, e, x_{i+1}, e, \cdots, e), f(e, \cdots, e), \cdots, f(e,\\
&\qquad \cdots, e))\\
&=f((x_1+\cdots+x_i), x_{i+1}, e, \cdots, e)\\
&=x_1+\cdots+x_i+x_{i+1}.
\end{aligned}$$

Thus, by induction, $f(x_1, x_2, \cdots, x_m)=x_1+x_2+\cdots+x_m$. Similarly $g(x_1, x_2, \cdots, x_n)=x_1+x_2+\cdots+x_n$.

Let $e$ be an idempotent element of $(A, f, g)$ and let $\sigma_i$ be the $i$-th translation defined in (1). For $x_1, x_2, \cdots, x_n$ in $A$,

$$\begin{aligned}
\sigma_i g(x_1, x_2, \cdots, x_n)&=f(e, \cdots, e, g(x_1, x_2, \cdots, x_n), e, \cdots, e)\\
&=f(g(e, \cdots, e), \cdots, g(x_1, x_2, \cdots, x_n), \cdots, g(e, \cdots, e))\\
&=g(f(e, \cdots, x_1, \cdots, e), f(e, \cdots, x_2, \cdots, e), \cdots, f(e, \cdots, x_n,\\
&\qquad \cdots, e))\\
&=g(\sigma_i x_1, \sigma_i x_2, \cdots, \sigma_i x_n).
\end{aligned}$$

Similarly, we have $\sigma_i f(x_1, x_2, \cdots, x_m)=f(\sigma_i x_1, \sigma_i x_2, \cdots, \sigma_i x_m)$. Thus, $\sigma_i$ is a homomorphism of $(A, f, g)$. By the same arguments, we can show each $\tau_i$ is a homomorphism. Clearly, $\sigma_i$ and $\tau_j$ are automorphisms if $e$ is regular. Now, for any $x$ in $A$,

$$\begin{aligned}
\sigma_i\tau_j x&=f(e, \cdots, e, g(e, \cdots, x, \cdots, e), e, \cdots, e)\\
&=f(g(e, \cdots, e), \cdots, g(e, \cdots, x, \cdots, e), \cdots, g(e, \cdots, e))\\
&=g(f(e, \cdots, e), \cdots, f(e, \cdots, x, \cdots, e), \cdots, f(e, \cdots, e))
\end{aligned}$$

$=\tau_j\sigma_i x$

and hence $\sigma_i\tau_j=\tau_j\sigma_i$. Similarly, $\sigma_i\sigma_j=\sigma_j\sigma_i$ and $\tau_i\tau_j=\tau_j\tau_i$ for every $i$ and $j$. By this we have proved:

LEMMA 4. *Let $(A,f,g)$ be a medial algebra and $e$ an idempotent element. Then the translations defined in (1) are endomorphisms and they commute pairwise. If, furthermore, $e$ is regular then they are automorphisms.*

LEMMA 5. *Let $(A,f,g)$ be a medial algebra and $\alpha_1,\cdots,\alpha_m,\beta_1,\cdots,\beta_n$ be pairwise commuting endomorphisms of $(A,f,g)$. Let $f^*$ and $g^*$ be operations on $A$ defined by*

*$f^*(x_1,\cdots,x_m)=f(\alpha_1x_1,\cdots,\alpha_mx_m)$ and $g^*(x_1,\cdots,x_n)=f(\beta_1x_1,\cdots,\beta_nx_n)$. Then $(A,f^*,g^*)$ is also a medial algebra. Furthermore, $\alpha_1,\cdots,\alpha_m,\beta_1,\cdots,\beta_n$ are endomorphisms of $(A,f^*,g^*)$.*

*Proof*. For $x_{ij}\in A$, $i=1,2,\cdots,m$, $j=1,2,\cdots,n$,

$$\begin{aligned}
&f^*(g^*(x_{11},\cdots,x_{1n}),\cdots,g^*(x_{m1},\cdots,x_{mn}))\\
&=f(g(\alpha_1\beta_1x_{11},\cdots,\alpha_1\beta_nx_{1n}),\cdots,g(\alpha_m\beta_1x_{m1},\cdots,\alpha_m\beta_nx_{mn}))\\
&=f(g(\beta_1\alpha_1x_{11},\cdots,\beta_n\alpha_1x_{1n}),\cdots,g(\beta_1\alpha_mx_{m1},\cdots,\beta_n\alpha_mx_{mn}))\\
&=g(f(\beta_1\alpha_1x_{11},\cdots,\beta_1\alpha_mx_{m1}),\cdots,g(\beta_n\alpha_1x_{1n},\cdots,\beta_n\alpha_mx_{mn}))\\
&=g^*(f^*(x_{11},\cdots,x_{m1}),\cdots,f^*(x_{1n},\cdots,x_{mn})).
\end{aligned}$$

Thus $f^*$ and $g^*$ commute. Similarly, $f^*$ and $g^*$ commute with themselves. Now,

$$\begin{aligned}
\alpha_ig^*(x_1,x_2,\cdots,x_n)&=\alpha_ig(\beta_1x_1,\beta_2x_2,\cdots,\beta_nx_n)=g(\alpha_i\beta_1x_1,\alpha_i\beta_2x_2,\cdots,\alpha_i\beta_nx_n)\\
&=g(\beta_1\alpha_ix_1,\beta_2\alpha_ix_2,\cdots,\beta_n\alpha_ix_n)=g^*(\alpha_ix_1,\alpha_ix_2,\cdots,\alpha_ix_n).
\end{aligned}$$

Similarly, $\alpha_if^*(x_1,x_2,\cdots,x_m)=f^*(\alpha_ix_1,\alpha_ix_2,\cdots,\alpha_ix_m)$. Hence, $\alpha_i$ is an endomorphism of $(A,f^*,g^*)$. By the same way, each $\beta_j$ is an endomorphism of $(A,f^*,g^*)$.

LEMMA 6. *Let $(A,f,g)$ be a medial algebra with a regular idempotent elemente. Define new operations $f^*$ and $g^*$ on $A$ by*

$$f^*(x_1,\cdots,x_m)=f(\sigma_1^{-1}x_1,\cdots,\sigma_m^{-1}x_m)\text{ and }g^*(x_1,\cdots,x_n)=f(\tau_1^{-1}x_1,\cdots,\tau_n^{-1}x_n)\qquad(3)$$

*where $\sigma_1,\cdots,\sigma_m$, $\tau_1,\cdots,\tau_n$ are translations defined in (1). Then $(A,f^*,g^*)$ is a medial algebra with $e$ as an identity element.*

*Proof*. By Lemma 4, $\sigma_1,\cdots,\sigma_m,\tau_1,\cdots,\tau_n$ are commuting automorphisms, and hence so are $\sigma_1^{-1}, \cdots, \sigma_m^{-1}, \tau_1^{-1}, \cdots, \tau_n^{-1}$. By Lemma 5, $(A, f^*, g^*)$ is a medial algebra. Now

$$f^*(e, \cdots, x, \cdots, e)=f(\sigma_1^{-1}e, \cdots, \sigma_i^{-1}x, \cdots, \sigma_m^{-1})=f(e, \cdots, \sigma_i^{-1}, \cdots, e)$$
$$=x.$$

Similarly, $g^*(e, \cdots, x, \cdots, e)=x$. Thus $e$ is an identity element of $(A, f^*, g^*)$.

THEOREM 7. *Let $(A, f, g)$ be a medial algebra with a regula idempotent element $e$. Then there is a commutative semigroup $(A, +)$ with $e$ as the identity element and pairwise commuting automorphisms $\sigma_1, \cdots, \sigma_m, \tau_1, \cdots, \tau_n$ of $(A, +)$ such that*

$$f(x_1, \cdots, x_m)=\sigma_1 x_1+\cdots+\sigma_m x_m \text{ and } g(x_1, \cdots, x_n)=\tau_1 x_1+\cdots+\tau_n x_n$$

*for all $x_1, \cdots, x_m, \cdots, x_n$ in $A$.* (4)

*Proof.* Define operations $f^*$ and $g^*$ as are in (3). $(A, f^*, g^*)$ is a medial algebra with an identity element $e$, by Lemma 6. Thus, by Theorem 3, there is a commutative semigroup $(A, +)$ with $e$ as the identity element such that

$$f^*(x_1, \cdots, x_m)=f(\sigma_1^{-1}x_1, \cdots, \sigma_m^{-1}x_m)=x_1+\cdots+x_m$$

and

$$g^*(x_1, x_2, \cdots, x_n)=g(\tau_1^{-1}x_1, \tau_2^{-1}x_2, \cdots, \tau_n^{-1}x_n)=x_1+x_2+\cdots+x_n.$$

Thus, (4) holds. By, Lemmas 4 and 5, $\sigma_1, \cdots, \sigma_m, \tau_1, \cdots, \tau_n$ are pairwise commuting automorphisms of $(A, f^*, g^*)$. As is in (2), '+' is defined by $x+y=f^*(x, y, e, \cdots, e)=g^*(x, y, e, \cdots, e)$. Thus,

$$\sigma_i(x+y)=\sigma_i f^*(x, y, e, \cdots, e)=f(\sigma_i x, \sigma_i y, \sigma_i e, \cdots, \sigma_i e)$$
$$=f^*(\sigma_i x, \sigma_i y, e, \cdots, e)=\sigma_i x+\sigma_i y.$$

That is, $\sigma_i$ is an automorphism of $(A, +)$ for each $i$. Similarly, $\tau_j$ is an automorphism of $(A, +)$ for each $j$.

## 3. Medial algebras without regular elements

LEMMA 8. *Let $(A, f, g)$ be a medial algebra, $\pi$ a permutation on $\{1, 2, \cdots, m\}$ and $\rho$ a permutation on $\{1, 2, \cdots, n\}$. Let $f^*$ and $g^*$ be operations on $A$ defined by*

$$f^*(x_1, \cdots, x_m)=f(x_{\pi 1}, \cdots, x_{\pi m}) \text{ and } g^*(x_1, \cdots, x_n)=g(x_{\rho 1}, \cdots, x_{\rho n}).$$

*Then, $(A, f^*, g^*)$ is also a medial algebra.*

*Proof*. We only show that $f^*$ and $g^*$ commute, and others can be proved similarly. For $x_{ij}\in A$,

$$\begin{aligned}
&f^*(g^*(x_{11},\cdots,x_{1n}),\cdots,g^*(x_{m1},\cdots,x_{mn}))\\
&=f(g(x_{\pi 1\rho 1},\cdots,x_{\pi 1\rho n}),\cdots,g(x_{\pi m\rho 1},\cdots,x_{\pi m\rho n}))\\
&=g(f(x_{\pi 1\rho 1},\cdots,x_{\pi m\rho 1}),\cdots,f(x_{\pi 1\rho n},\cdots,x_{\pi m\rho n}))\\
&=g^*(f^*(x_{11},\cdots,x_{m1}),\cdots,f^*(x_{1n},\cdots,x_{mn})),
\end{aligned}$$

as is wanted.

THEOREM 9. *Let $(A,f,g)$ be a medial algebra with an idempotent element $e$ which is $i$– and $j$–regular with respect to both $f$ and $g$ for fixed $i$ and $j$ $(i\neq j)$, Then there is a commutative semigroup $(A,+)$ with $e$ as the identity element and pairwise commuting endomorphisms $\sigma_1,\cdots,\sigma_m,\tau_1,\cdots,\tau_n$ of $(A,+)$ such that*

$$f(x_1,\cdots,x_m)=\sigma_1x_1+\cdots+\sigma_mx_m \text{ and } g(x_1,\cdots,x_n)=r_1x_1+\cdots+r_nx_n$$

*for all $x_1,\cdots,x_m,\cdots,x_n$ in $A$. Furthermore, $\sigma_i,\sigma_j,\tau_i,\tau_j$ are automorphisms.*

*Proof*. Due to the preceding lemma, we may assume that $e$ is 1– and 2–regular. Let $\sigma_1,\cdots,\sigma_m,\tau_1,\cdots,\tau_n$ be the translations defined in (1). Then, they are pairwise commuting endomorphisms of $(A,f,g)$ by Lemma 4 and $\sigma_1,\sigma_2,\tau_1,\tau_2$ are automorphisms. With the definition (1) of these mappings in mind, we have

$$\begin{aligned}
&f(\sigma_1^{-1}x,\sigma_2^{-1}y,e,\cdots,e)\\
&=f(g(\tau_1^{-1}\sigma_1^{-1}x,e,\cdots,e),g(e,\tau_2^{-1}\sigma_2^{-1}y,e,\cdots,e),g(e,\cdots,e),\cdots,g(e,\\
&\quad\cdots,e))\\
&=g(f(\tau_1^{-1}\sigma_1^{-1}x,e,\cdots,e),f(e,\tau_2^{-1}\sigma_2^{-1}y,e,\cdots,e),f(e,\cdots,e),\cdots,f(e,\\
&\quad\cdots,e))\\
&=g(f(\sigma_1^{-1}\tau^{-1}x,e,\cdots,e),f(e,\sigma_2^{-1}\tau_2^{-1}y,e,\cdots,e),e,\cdots,e)\\
&=g(\tau_1^{-1}x,\tau_2^{-1}y,e,\cdots,e)
\end{aligned}$$

for all $x,y\in A$. With this, we define a binary operation '$+$' on $A$ by

$$x+y=f(\sigma_1^{-1}x,\sigma_2^{-1}y,e,\cdots,e)=g(\tau_1^{-1}x,\tau_2^{-1}y,e,\cdots,e). \quad (5)$$

One can easily verify that $(A,+)$ is a medial groupoid with $e$ as the identity element. Thus, $(A,+)$ is a commutative semigroup by Lemma 2. Furthermore, it can be seen that $\sigma_1,\cdots,\sigma_m,\tau_1,\cdots,\tau_n$ are endomorp hisms of $(A,+)$. From (5), $f(x_1,x_2,e,\cdots,e)=\sigma_1x_1+\sigma_2x_2$. Suppose $f(x_1,\cdots,x_i,e,\cdots,e)=\sigma_1x_1+\cdots+\sigma_ix_i$, then

$$\begin{aligned}
&f(x_1, \cdots, x_i, x_{i+1}, e, \cdots, e) \\
&=f(f(\sigma_1^{-1}x_1, e, \cdots, e), \cdots, f(\sigma_1^{-1}x_i, e, \cdots, e), f(e, \sigma_2^{-1}x_{i+1}, e, \cdots, e), \cdots \\
&\quad , f(e, \cdots, e)) \\
&=f(f(\sigma_1^{-1}x_1, \cdots, \sigma_1^{-1}x_i, e, \cdots, e), f(e, \cdots, e, \sigma_2^{-1}x_{i+1}, e, \cdots, e), \cdots, f \\
&\quad (e, \cdots, e)) \\
&=f(\sigma_1^{-1}f(x_1, \cdots, x_i, e, \cdots, e), \sigma_2^{-1}f(e, \cdots, e, x_{i+1}, e, \cdots, e), \cdots, e) \\
&=f(x_1, \cdots, x_i, e, \cdots, e)+f(e, \cdots, e, x_{i+1}, e, \cdots, e) \\
&=\sigma_1x_1+\cdots+\sigma_ix_i+\sigma_{i+1}x_{i+1}.
\end{aligned}$$

Thus, by induction, $f(x_1, x_2, \cdots, x_m)=\sigma_1x_1+\cdots+\sigma_mx_m$. Simlarly $g(x_1, x_2, \cdots, x_n)=\tau_1x_1+\cdots+\tau_nx_n$.

## 4. Closing

For a medial algebra $(A, \Omega)$ with many operations, if we assume the relevant properties of an element with respect to every operation in $\Omega$, then we can get the similar result as before.

For groupoids, the existence of a regular elements (not being idempotent) is sufficient for the operation of a medial groupoid to be defined on a commutative monoid by translating the operation obtained in Theorem 7 ([3], [6]). That is, for any medial groupoid $(G, \cdot)$ with a regular element, there is a commutative monoid $(G, +)$, two automorphisms $\alpha$ and $\beta$ of $G(, +)$, and an element $d$ of $G$ such that $x \cdot y = ax+\beta y+d$ for all $x, y$ in $G$. Our question is how we do this kind of work for medial algebras with a regular element which is not idempotent. Is it always possible to represent a medial algebra with a regular element in this way?

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