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HYPERBOLICITY IN THE BEURLING'S GENERALIZED DISTRIBUTION SPACES

DAE-HYEON PAHK AND BU-HYEON KANG

1. Introduction

In [4], L. Gårding showed that, in the distribution space, the (unique) existence of a fundamental solution of $P(D)$, supported in a proper cone of $H=\{x\in \boldsymbol{R}^n : \langle x, N\rangle \geqq 0\}$, is equivalent to the existence of a constant $c>0$ such that

(2) $P_m(N)\neq 0$ and
$P(\xi+i\tau N)\neq 0$ for every $\tau<-c$ and $\xi\in \boldsymbol{R}^n$.

Later E. Larsson extended this result to the generalized distribution spaces of Gevrey classes in [7]. He showed that, in those spaces, the existence of a fundamental solution of $P(D)$, supported in a proper cone of H, is equivalent to the existence of a constant $c>0$ such that

(3) $P_m(N)\neq 0$ and
$P(\xi+i\tau N)\neq 0$ when $\tau<-c(1+|\xi|^{1/d})$ and $\xi\in \boldsymbol{R}^n$.

In this paper we consider this problem in the generalized distribution spaces, defined by Beurling, which include both spaces above as special cases. Our results unify (2) and (3) in a general setting and contain both results as special cases by taking $\omega(\xi)=\log(1+|\xi|)$ and $\omega(\xi)=|\xi|^{1/d}$, $d>1$, respectively.

For completeness we briefly review the generalized distribution spaces and their properties which we need in this paper. Further details are given in [2]. We denote by M_c the set of all continuous real-valued functions ω on $\boldsymbol{R}^n$ satisfying the following facts:

(α) $0=\omega(0)\leq\omega(\xi+\eta)\leq\omega(\xi)+\omega(\eta)$, $\xi,\eta\in \boldsymbol{R}^n$.

(β) $\int_{\boldsymbol{R}^n}\frac{\omega(\xi)}{(1+|\xi|)^{n+1}}d\xi<\infty$.

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(γ) $\omega(\xi)\geq a + b\log(1+|\xi|)$ for some constants a and $b>0$.

(δ) $\omega(\xi)=\Omega(|\xi|)$ for some concave function Ω on $[0,\infty)$.

For example, $\omega(\xi)=\log(1+|\xi|)^{1/d}$, $d>1$, satisfy all the conditions. Throughout this paper ω represents an element in M_c, Ω the concave function in (δ), and U an open set in $\boldsymbol{R}^n$.

Let $D_\omega(U)$ be the set of all $\phi\in L^1(\boldsymbol{R}^n)$ such that ϕ has compact support in U and

$$\|\phi\|_\lambda=\int_{\boldsymbol{R}^n}|\hat{\phi}(\xi)|e^{\lambda\omega(\xi)}d\xi<\infty \text{ for every } \lambda>0.$$

The topology on this space is given by the inductive limit topology of the Fréchet spaces $D_\omega(K)$ induced by the above semi-norms where K is a compact set in U. We denote by $E_\omega(U)$ the set of all complex valued functions ψ in U such that $\phi\psi$ is in $D_\omega(U)$ for every $\phi\in D_\omega(U)$, equipped with the topology generated by the semi-norms $\|\phi\psi\|_\lambda$ for every $\phi\in D_\omega(U)$ and $\lambda>0$. The dual space of $D_\omega(U)$ is denoted by $D_\omega'(U)$ whose elements are called generalized distributions on U. The dual space $E_\omega'(U)$ of $E_\omega(U)$ can be identified with the set of all elements of $D_\omega'(U)$ which have compact support in U. The following two results will be frequently used in the next sections. Their proofs can be found in [2].

Lemma 1.1 *$\omega(\xi)=o(|\xi|/\log|\xi|)$ as $|\xi|\to\infty$ for all $\omega\in M_c$.*

Lemma 1.2 (Paley-Wiener type theorem). *Let K be a compact convex set in $\boldsymbol{R}^n$ with support function S. If F is an entire analytic function of n complex variables $\zeta=\xi+i\eta=(\zeta_1,\dots,\zeta_n)$, the following statements are equivalent:*

(*a*) *For each $\lambda>0$ and each $\varepsilon>0$ there is a constant $c_{\lambda,\varepsilon}$ such that*

$$\int_{\boldsymbol{R}^n}|F(\xi+i\eta)|e^{\lambda\omega(\xi)}d\xi\leq c_{\lambda,\varepsilon}e^{S(\eta)+\varepsilon|\eta|},\ \eta\in\boldsymbol{R}^n.$$

(*b*) *For each $\lambda>0$ and each $\varepsilon>0$ there is a constant $c_{\lambda,\varepsilon}$ such that*
$|F(\xi+i\eta)|\leq c_{\lambda,\varepsilon}e^{S(\eta)+\varepsilon|\eta|-\lambda\omega(\xi)},\ \zeta\in\boldsymbol{C}^n.$

(*c*) $F(\zeta)=\int_{\boldsymbol{R}^n}e^{-i\langle x,\zeta\rangle}\phi(x)\,dx$ *for some* $\phi\in D_\omega(K)$.

2. Necessity of ω-hyperbolicity

Let H be the half-space $\langle x, N\rangle\geq 0$ and $E_\omega(H)$ the set of all functions in $E_\omega(\boldsymbol{R}^n)$ which have their supports in H. We denote by

$$|\eta|_N=\inf_{t\in R}|\eta-tN| \text{ for } \eta\in\boldsymbol{R}^n.$$

LEMMA 2.1. *There is a function $\chi\in E_\omega(H)$ whose value is 1 on the set $\Gamma_1=\{x\in\boldsymbol{R}^n : \langle x, N\rangle\geq\frac{1}{2}\}$ and 0 on $\Gamma_2=\{x\in\boldsymbol{R}^n : \langle x, N\rangle\leq\frac{1}{4}\}$.*

Proof. Consider a locally integrable function $u(x)$ which is 1 on $\Gamma_1+B(0,\frac{\delta}{4})$ and 0 otherwise, where δ is the distance between Γ_1 and Γ_2. Then u is in D_ω' and, from Theorem 1.7.3 [2], $u*\phi\in E_\omega$ for any ϕ in D_ω. Taking $\phi\in D_\omega(B(0,\frac{\delta}{4}))$ such that $\int\phi(x)dx=1$, it is easy to see that $u*\phi$ satisfies all the requirements.

We assume that $P(D)$ is a differential operator of order m with constant coefficients, say,

$$P(D)=\sum_{|\alpha|\leq m}a_\alpha D^\alpha$$

where $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, $D_j=\frac{1}{i}\frac{\partial}{\partial x_j}$ and $\alpha=(\alpha_1,\ldots,\alpha_n)$, and $|N|=1$.

THEOREM 2.1. *Assume that the equation $P(D)\phi=\psi$ has a unique solution $\phi\in E_\omega(H)$ for every $\psi\in E_\omega(H)$. Then there is a constant $c>0$ such that*

$$(4)\qquad P(\zeta)=P(\xi+i\eta)\neq0 \text{ if } \langle\eta, N\rangle<-c(1+|\eta|_N+\omega(\xi)) \text{ and } \zeta=\xi+i\eta\in\boldsymbol{C}^n.$$

Proof. Since $E_\omega(H)$ is a Fréchet space and the linear mapping $P(D)\phi$ is continuous from $E_\omega(H)$ onto itself, due to Hahn-Banach Theorem, the inverse mapping is also continuous. This gives that, for all $\lambda>0$ and $\phi\in D_\omega$, there are positive constants c, λ_0 and a function $\phi_0\in D_\omega$ such that

$$\|\phi\psi\|_\lambda\leq c\|\phi_0P(D)\psi\|_{\lambda_0} \text{ for every } \psi\in E_\omega(H).$$

Applying this inequality for $\phi\in D_\omega$ with $\phi(N)=1$ and $\psi=\chi e^{i\langle x-N,\zeta\rangle}$, where χ is the function in Lemma 2.1, we get

$$(5)\qquad 1=|\phi(N)\psi(N)|\leq(2\pi)^{-n}\int|\hat{\phi\psi}|d\xi \leq(2\pi)^{-n}\|\phi\psi\|_1$$

$$\leq c\|\phi_0 P(D)[\exp(i\langle x-N,\zeta\rangle)\chi(x)]\|_{\lambda_0}$$

When $P(\zeta)=0$, from Leibniz formula we have

$$\phi_0(x)P(D)e^{i\langle x-N,\zeta\rangle}\chi(x)=\sum_{\gamma\neq 0}\frac{1}{\gamma!}P^{(\gamma)}(\zeta)e^{i\langle x-N,\zeta\rangle}\phi_0(x)D^\gamma\chi(x).$$

The support of $g_\gamma(x)=\phi_0(x)D^\gamma\chi(x)$ is contained in a bounded set B of $\{x:\frac{1}{4}\leq\langle x,N\rangle\leq\frac{1}{2}\}$ when $\gamma\neq 0$. According to Lemma 1.2, for λ_0 and $\varepsilon>0$, there is a constant $c_{\lambda_0,\varepsilon}$ such that, for every $\alpha\in\boldsymbol{R}^n$,

$$|\int e^{-i\langle x,\alpha\rangle}g_\gamma(x)e^{i\langle x-N,\zeta\rangle}dx|$$
$$=e^{\langle\eta,N\rangle}|\hat{g}_\gamma(\alpha-\zeta)|$$
$$\leq c_{\lambda_0,\varepsilon}\exp[\langle\eta,N\rangle+S(-\eta)+\varepsilon|\eta|+\lambda_0\omega(\xi-\alpha)]$$

where S is the support function of the set B. Estimating (5) with this fact there is a polynomial Q of degree less than m such that

$$(6)\qquad 1\leq Q(|\zeta|)\exp\ [\langle\eta,N\rangle+S(-\eta)+\varepsilon|\eta|+\lambda_0\omega(\xi)]$$

because the constant λ_0 can be taken large enough. In order to estimate $S(-\eta)$ we write $x=tN+y$ where $\langle y,N\rangle=0$. This gives that, for $x\in B$, $\frac{1}{4}\leq t\leq\frac{1}{2}$ and $|y|\leq D$ for some fixed constant D. When $\langle\eta,N\rangle<0$, we obtain

$$S(-\eta)=\sup_{x\in B}\langle x,-\eta\rangle\leq\sup_{\frac{1}{4}\leq t\leq\frac{1}{2}}t\langle N,-\eta\rangle+\sup_{|y|\leq D}\langle y,-\eta\rangle$$
$$\leq-\frac{1}{2}\langle\eta,N\rangle+D\inf_t|\eta-tN|.$$

Substituting this inequality in (6) we have

$$1\leq Q(|\zeta|)\exp[\frac{1}{2}\langle\eta,N\rangle+\varepsilon|\eta|+D|\eta|_N+\lambda_0\omega(\xi)].$$

Taking logarithm in both sides we get, for some large constants M and C,

$$0\leq\log\ Q(|\zeta|)+\frac{1}{2}\langle\eta,N\rangle+\varepsilon|\eta|+D|\eta|_N+\lambda_0\omega(\xi)$$
$$\leq M\log(1+|\xi|+|\eta|)+\frac{1}{2}\langle\eta,N\rangle+\varepsilon|\eta|+C'(1+|\eta|_N+\omega(\xi))$$
$$\leq\frac{1}{4}\langle\eta,N\rangle+2C'(1+|\eta|_N+\omega(\xi))$$
$$+M\log(1+|\xi|+|\langle\eta,N\rangle|+|\eta|_N)-\frac{1}{4}|\langle\eta,N\rangle|+\varepsilon|\langle\eta,N\rangle|$$
$$+\varepsilon|\eta|_N-C'(1+|\eta|_N+\omega(\xi))$$

since $|\eta|\leq|\langle\eta,N\rangle|+|\eta|_N$. Choosing $\varepsilon=1/8$, we get

$$0\leq\frac{1}{4}\langle\eta, N\rangle+2C'(1+|\eta|_N+\omega(\xi))+M\log(1+|\xi|)$$
$$+M\log(1+|\langle\eta, N\rangle|)+M\log(1+|\eta|_N)-\frac{1}{8}|\langle\eta, N\rangle|$$
$$+(\frac{1}{8}-C')|\eta|_N-C'(1+\omega(\xi)).$$

By condition (γ), $\log(1+|\xi|)\leq\omega(\xi)/b-a/b$. Choosing C_0 for which $t\geq C_0$ implies that $\exp(t/8M)\geq 1+t$, we have $(1/8)|\langle\eta, N\rangle|\geq M\log(1+|\langle\eta, N\rangle|)$ when $\langle\eta, N\rangle<-C_0$. Choosing $C'\geq M+\frac{1}{8}$, we get

$$M\log(1+|\eta|_N)+(\frac{1}{8}-C')|\eta|_N\leq 0 \text{ for all } |\eta|_N.$$

Hence, if we choose $C=\max\{C_0, M/b, (-aM)/b, M+\frac{1}{8}\}$, we have

$$P(\zeta)=0 \text{ and } \langle\eta, N\rangle<-C_0 \Longrightarrow 0\leq\frac{1}{4}\langle\eta, N\rangle+2C(1+|\eta|_N+\omega(\xi)).$$

Therefore, for $\zeta=\xi+i\eta$, $\langle\eta, N\rangle<-8C(1+|\eta|_N+\omega(\xi))$ implies that $\langle\eta, N\rangle<-C_0$ and $P(\zeta)\neq 0$.

THEOREM 2.2. *$P_m(N)\neq 0$ if there is a constant c such that $P(\xi+i\eta)\neq 0$ when $\langle\eta, N\rangle<-C(1+|\eta|_N+\omega(\xi))$.*

Proof. By means of a rotation we may think $N=(1, 0, \ldots, 0)$. Now assume that $P_m(N)=0$. Since P_m is not identically zero, there are constants $(\alpha_j)_{j=2}^n$ such that $P_m(1, \alpha_2, \ldots, \alpha_n)\neq 0$. We consider the following polynomial

$$Q(\lambda, \mu)=P(\lambda, \lambda\mu\alpha_2, \ldots, \lambda\mu\alpha_n)=\prod_{\nu=0}^{m}\lambda^\nu R_\nu(\mu)$$

where $R_m(\mu)=P_m(1, \mu\alpha_2, \ldots, \mu\alpha_n)$ is not identically zero, due to the choice of (α_j). By the hypothesis, the zeros $\lambda(\mu)$ of $Q(\lambda, \mu)$ satisfy

(7) $\operatorname{Im}\lambda(\mu)>-c(1+|\operatorname{Im}\lambda(\mu)\mu|+\omega(\operatorname{Re}\lambda(\mu)(1, \mu\alpha_2, \ldots, \mu\alpha_n)))$

for a suitable constant c. Since $R_m(\mu)$ is not identically zero, the zeros can be developed into a Puiseux series around $\mu=0$. Hence we obtain

$$Q(\lambda, \mu)=R_m(\mu)\prod_{j=1}^{m}(\lambda-\lambda_j(\mu))$$

where every $\lambda_j(\mu)$ is an analytic function of $\mu^{1/p}$ when $0<|\mu|<\delta$, for some positive integer p, without any essential singularity at $\mu^{1/p}=0$. That is,

$$\lambda_j(\mu)=\sum_{k=N_j}^{\infty} a_k(\mu^{1/p})^k$$

where N_j is an integer. Because of (7) and $R_m(0)=P_m(N)=0$, at least one $R_\nu(0)\neq 0$ for $0<\nu<m$ and so at least one quotient $R_\nu(\mu)/R_m(\mu)$ tends to infinity as $\mu\to 0$. Consequently, $|\lambda_{j_0}(\mu)|\to\infty$ for some j_0 as $\mu\to 0$ and so $N_{j_0}=N$ is a negative integer. Thus $\lambda_{j_0}(\mu)$ behaves asymptotically as $a_N(\mu^{1/p})^N$ when $\mu\to 0$, hence

(8) $\qquad \lambda_{j_0}(\mu)=a_N(\mu^{1/p})^N(1+o(1))$ as $\mu\to 0$.

Applying (7) for $\lambda_{j_0}(\mu)$ we have

$$|\operatorname{Im}\lambda_{j_0}(\mu)|\leq c(1+|\operatorname{Im}\lambda_{j_0}(\mu)\mu|+\omega(\operatorname{Re}\lambda_{j_0}(\mu)(1,\mu\alpha_2,\ldots,\mu\alpha_n))).$$

But, considering the growth of both sides with (8) and Lemma 1.1 the left side tends to infinity faster than the right side when μ tends to 0, which is a contradiction.

The relation (4) is reduced to

(9) $\qquad P(\xi+i\tau N)\neq 0$ when $\xi\in\boldsymbol{R}^n$ and $\tau<-c(1+\omega(\xi))$

in the special case $\eta=\tau N$, $\tau\in\boldsymbol{R}$. From this fact we get

Definition 2.1. A polynomial P is called ω-hyperbolic with respect to N if $P_m(N)\neq 0$ and P satisfies (9) for some constant c.

Lemma 2.2. *If $E\in D_\omega'$ with support in a proper cone of H and ψ is in $E_\omega(H)$, then $E*\psi$ is in $E_\omega(H)$.*

Proof. Let $x\in\boldsymbol{R}^n$, then supp $E(y)\cap$ supp $\psi(x-y)$ is compact. Hence, if $\alpha(y)\in D_\omega$ is a local unit for this compact set, then we can define the convolution $E*\psi(x)$ by $E*\psi(x)=\langle E(y),\alpha(y)\psi(x-y)\rangle$. It is then clear that it is well-defined. Note that we need only one local unit when x varies on a compact subset of $\boldsymbol{R}^n$. Let $\phi\in D_\omega$ and set $v(x)=E*\psi(x)\phi(x)$. Then, for some local unit α, we get

$$\begin{aligned} v(x)&=\phi(x)\langle E(y),\ \alpha(y)\psi(x-y)\rangle\\ &=\phi(x)\langle E(y)\alpha(y),\ \psi(x-y)\rangle\\ &=\phi(x)\langle u(y),\ \psi(x-y)\rangle \end{aligned}$$

where $u(y)=E(y)\phi(y)\in E_\omega'(H)$. Let $A=\max\{\|x\| : x\in\text{supp }\phi\}$ and $B=\max\{\|y\| : y\in\text{supp } u\}$. Then, for all $x\in$ supp ϕ and $y\in$ supp u we have $x-y\in B(0,A+B)$. Let $\beta(y)$ be a local unit for this set, then

$$\begin{aligned} v(x)&=\phi(x)\langle u(y),\ \psi(x-y)\rangle\\ &=\phi(x)\ \langle u(y),\ \beta\psi(x-y)\rangle+\phi(x)\langle u(y), \end{aligned}$$

$(1-\beta(x-y))\phi(x-y)>$.

Since the last is 0, we have $v(x)=\phi(x)[E*\beta\phi](x)$, which is a member of D_ω by Theorem 1.7.3 in [2].

COROLLARY 2.3. *If $u\in E_\omega'$ and $\phi\in E_\omega$, then $u*\phi\in E_\omega$.*

Proof. See the proof of the preceding theorem.

THEOREM 2.3. *If $P(D)$ has a fundamental solution $E\in D_\omega'$ whose support is contained in a proper cone of H, then P is ω-hyperbolic with respect to N.*

Proof. Since $E*\phi\in E_\omega(H)$ and $P(D)(E*\phi)=\phi$ for all ϕ in $E_\omega(H)$, the mapping $P(D)\phi$ is surjective from $E_\omega(H)$ into itself. This mapping is also injective because $\phi=\delta*\phi=P(D)E*\phi=E*P(D)\phi=0$ when $P(D)\phi=0$. Thus, the result follows from Theorems 2.1 and 2.2.

3. Sufficiency of ω-hyperbolicity

We now prove that the ω-hyperbolicity of $P(D)$ with respect to N implies the existence of the unique fundamental solution in D_ω' whose support is contained in a proper cone of H. To do this we study some algebraic properties of ω-hyperbolic polynomials.

LEMMA 3.1. *For any compact set K of $\{\theta\in \mathbf{R}^n : P_m(\theta)\neq 0\}$ there is a constant c such that, for every $\theta\in K$, $P(\zeta+\tau\theta)\neq 0$ when $\tau\in\mathbf{C}, \zeta\in\mathbf{C}^n$, and $|\tau|>c(1+|\zeta|)$.*

Proof. We can write

$$P(\zeta+\tau\theta)=P_m(\theta)\tau^m+\sum_{\nu=0}^{m-1}P_\nu(\zeta)\tau^\nu$$

where $P_\nu(\zeta)$ is a polynomial in ζ of degree $\leq m-\nu$ whose coefficients are depending on the given θ. Since θ varies on the compact set K, there is a constant c so large that

$$|P_\nu(\zeta)|\leq\frac{1}{2}[\min_{\theta\in K}|P_m(\theta)|]\ [\frac{1}{2}c(1+|\zeta|)]^{m-\nu}$$

for all $\nu=0, 1, \ldots, m-1$. This implies that, for $|\tau|<c(1+|\zeta|)$,

$$|P(\zeta+\tau\theta)|\geq|P_m(\theta)||\tau|^m-\sum_{\nu=0}^{m-1}|P_\nu(\zeta)||\tau|^\nu$$

$$\geq |P_m(\theta)||\tau|^m - \frac{1}{2}[\min_{\theta\in K}|P_m(\theta)|]\sum_{\nu=0}^{m-1}[\frac{1}{2}c(1+|\zeta|)]^{m-\nu}|\tau|^\nu$$

$$\geq \frac{1}{2}|P_m(\theta)||\tau|^m > 0 \text{ for every } \theta\in K.$$

From this lemma and the condition of ω, we immediately have that the inequality (9) is equivalent to the following:

(10) There exists a constant c such that $P(\xi+i\tau N)\neq 0$ for $\mathrm{Re}\,\tau < -c(1+\omega(\xi))$ when $\tau\in\boldsymbol{C}$ and $\xi\in\boldsymbol{R}^n$.

THEOREM 3.2. *P is ω-hyperbolic with respect to $-N$ if P is ω-hyperbolic with respect to N.*

Proof. The homogeneity of the principal part P_m gives that $P_m(-N)=(-1)^m P_m(N)\neq 0$. It follows from (9) that all the roots of $P(\xi+i\tau N)=0$ satisfy Re $\tau\geq -c(1+\omega(\xi))$ for some constant c when $\xi\in\boldsymbol{R}^n$. Denoting the zeros of $P(\xi+i\tau N)$ by τ_j, the zeros of the polynomial $P(\xi-i\tau N)$ are $-\tau_j$. Since the coefficients of τ^m and τ^{m-1} of $P(\xi+i\tau N)$ are $i^m P_m(N)\neq 0$ and a linear function of $\xi\in\boldsymbol{R}^n$ respectively, $\sum_{j=1}^m \tau_j$ is a linear function of ξ. This implies that $\sum_{j=1}^m \mathrm{Re}\tau_j$ is also a linear function of ξ bounded below by $-mc(1+\omega(\xi))$. According to the Lemma 1.1, $\sum_{j=1}^m \mathrm{Re}\,\tau_j$ then must be a constant L. This gives $-\mathrm{Re}\tau_k= -L+\sum_{j\neq k}\mathrm{Re}\tau_j \geqq -L-(m-1)c(1+\omega(\xi))$. Therefore, when $\xi\in\boldsymbol{R}^n$ and $\mathrm{Re}\tau < -c(1+\omega(\xi))$ for some constant c, $P(\xi+i\tau(-N))\neq 0$.

From the proof of Theorem 3.2 we have

COROLLARY 3.3. *If P is ω-hyperbolic with respect to N, there is a constant c such that*

$$|\mathrm{Re}\tau| < c(1+\omega(\xi)) \text{ when } \xi\in\boldsymbol{R}^n \text{ and } P(\xi+i\tau N)=0.$$

THEOREM 3.4. *If P is ω-hyperbolic with respect to N, then P_m is also ω-hyperbolic with respect to N.*

Proof. We have $\lambda^{-m}P(\lambda\xi+i\lambda\tau N)\longrightarrow P_m(\xi+i\tau N)$ as $\lambda\to\infty$.
From the Corollary 3.3, the zeros in τ of the left-hand side are located

within the strip $|\mathrm{Re}\,\tau|\leqq\frac{c}{\lambda}(1+\omega(\lambda\xi))$. It follows from Lemma 1.1 that the zeros of $P_m(\xi+i\tau N)$ have zero real parts.

COROLLARY 3.5. *A homogeneous polynomial P is ω-hyperbolic with respect to N if and only if $P(N)\neq 0$ and the zeros of $P(\xi+\tau N)$ are real when $\xi\in\boldsymbol{R}^n$.*

We denote by $\Gamma(P,N)$ the set of all real vectors θ such that $P_m(\theta+\tau N)$ has only negative zeros. Then from Lemma 5.5.1 in [5] we have

LEMMA 3.6. *$\Gamma(P,N)$ is the connected component of N in the open set $\{\theta\in\boldsymbol{R}^n : P_m(\theta)\neq 0\}$ and a convex set.*

THEOREM 3.7. *Suppose P is ω-hyperbolic in the direction N and θ is in $\Gamma(P,N)$. Then there is a constant c such that*
$P(\xi+i\tau N+i\sigma\theta)\neq 0$ when $\xi\in\boldsymbol{R}^n$, $Re\,\sigma\leqq 0$ and $\tau\leqq -c(1+\omega(\xi))$.

Proof. For $\mathrm{Re}\,\sigma=0$, ω-hyperbolicity gives that, for some constant c, $|\tau|\leqq c(1+\omega(\xi)+\Omega(|\sigma\theta|))$ when $P(\xi+i\tau N+i\sigma\theta)=0$ and $\tau\in\boldsymbol{R}$, $\xi\in\boldsymbol{R}^n$. On the other hand, since $P_m(\theta)\neq 0$, Lemma 3.1 gives a constant D such that $|\sigma|\leqq D(1+|\xi|+|\tau|)$ when $P(\xi+i\tau N+i\sigma\theta)=0$. According to the concavity of Ω, we have

$$\Omega(|\sigma\theta|)\leqq\Omega[D(1+|\xi|+|\tau|)|\theta|]\leqq M[1+\Omega(|\xi|)+\Omega(|\tau|)]$$

for some constant M. Using this inequality and the property (δ) and Lemma 1.1, we can choose a constant c such that $|\tau|\leqq c(1+\omega(\xi))$ when $\xi\in\boldsymbol{R}^n$, $P(\xi+i\tau N+i\sigma\theta)=0$ and $\mathrm{Re}\,\sigma=0$.

For the case $\mathrm{Re}\,\sigma<0$ we look at $P(\xi+i\tau N+i\sigma\theta)$ as a polynomial in σ when $\xi\in\boldsymbol{R}^n$ and τ varies in $\tau\leqq -c(1+\omega(\xi))$. Here c is the constant obtained in the case $\mathrm{Re}\,\sigma=0$. Since the zeros of $P(\xi+i\tau N+i\sigma\theta)$ vary continuously with τ and $P(\xi+i\tau N+i\sigma\theta)$ has no zeros when $\xi\in\boldsymbol{R}^n$, $\mathrm{Re}\,\sigma=0$ and $\tau<-c(1+\omega(\xi))$, the numer of zeros σ with negative real part is constant when $\tau<-c(1+\omega(\xi))$. Hence it suffices to show that there is no zero σ with $\mathrm{Re}\,\sigma<0$ when $|\tau|$ is sufficiently large. Setting $\sigma=\mu\tau$, $P(\xi+i\tau N+i\sigma\theta)=0$ can be written in the form $i^{-m}\tau^{-m}P(\xi+i\tau(N+\mu\theta))=0$, which converges to $P_m(N+\mu\theta)=0$ as $\tau\to-\infty$. Since the zeros μ of $i^{-m}\tau^{-m}P(\xi+i\tau(N+\mu\theta))$ depend continuously on τ^{-1} and all the zeros of $P_m(N+\mu\theta)$ are negative, all the zeros of $P(\xi+i\tau N+i\sigma\theta)$ must have positive real parts when $\xi\in \boldsymbol{R}^n$ and $\tau\to-\infty$.

THEOREM 3.8. *If P is ω-hyperbolic with respect to N, then it is ω-hyperbolic in any direction of* $\Gamma(P, N)$.

Proof. Let $\theta\in\Gamma(P,N)$. We start by showing that P is ω-hyperbolic in the direction of $\theta+\varepsilon N$ with $\varepsilon>0$. In fact, from Theorem 3.7 with $\sigma=\varepsilon\tau$, we have

$$P(\xi+i\tau(\theta+\varepsilon N))\neq 0 \text{ for } \xi\in\boldsymbol{R}^n,\ \tau<-c(1+\omega(\xi))$$

and $P_m(\theta+\varepsilon N)\neq 0$ for sufficiently small $\varepsilon>0$, since $\Gamma(P,N)$ is open. Writing $\theta=(\theta-\varepsilon N)+\varepsilon N$, we then deduce from above that P is ω-hyperbolic with respect to θ, taking $\varepsilon>0$ small enough to ensure that $\theta-\varepsilon N$ remains in the open set $\Gamma(P,N)$.

We are now in a position to prove the main result.

THEOREM 3.9. *Suppose P is a ω-hyperbolic operator in the direction of N. Then P admits a unique fundamental solution whose support is contained in the dual cone*

$$\Gamma^*(P,N)=\{x\in\boldsymbol{R}^n : \langle x,\theta\rangle\geqq 0 \text{ for every } \theta\in\Gamma(P,N)\},$$

but in no smaller convex cone with vertex at 0.

Proof. The uniqueness follows from the fact that $P(D)u=0$ for $u\in D_\omega'$ with supp $u\subset H$ implies that $u\equiv 0$. If we write

$$(11)\quad P(\xi+i\tau\theta)=i^m P_m(\theta)\prod_{k=1}^{m}(\tau-\tau_k(\xi,\theta)) \text{ for } \theta\in\Gamma(P,N),$$

the ω-hyperbolicity of P in the direction θ assures the existence of a constant $c(\theta)$ such that

$$\operatorname{Re}\tau_k(\xi,\theta)\geqq -c(1+\omega(\xi)) \text{ for } \xi\in\boldsymbol{R}^n.$$

Taking τ in (11) to $t(1+\omega(\xi))$ with $t<-2c(\theta)$, we have

$$(12)\qquad |P(\xi+i\tau\theta)|\geqq|P_m(\theta)|\,|\tfrac{1}{2}t|^m(1+\omega(\xi))$$

for $\xi\in\boldsymbol{R}^n$. For such τ, letting $\sigma(\theta,t)$ be the surface $(\xi_1+i\tau\theta_1,\cdots,\xi_n+i\tau\theta_n)$ in $\boldsymbol{C}^n$, we define E on D_ω by

$$(13)\qquad \tilde{E}(\phi)=(2\pi)^{-n}\int_{\sigma(\theta,t)}\frac{\tilde{\phi}(\zeta)}{P(\zeta)}d\zeta \text{ for } \phi\in D_\omega(\boldsymbol{R}^n)$$

where we use the notations $\check{\phi}(x)=\phi(-x)$ and $\tilde{E}(\phi)=E(\check{\phi})$. Then the Paley–Wiener type theorem applied to ϕ implies that, for each $\lambda>0$ and $\varepsilon>0$, there is a constant $c_{\lambda,\varepsilon}$ such that

(14) $$|\hat{\check{\phi}}(\zeta)|\leqq c_{\lambda,\varepsilon}\|\phi\|_{\lambda}e^{t(1+\omega(\xi))S(\theta)+\varepsilon|t(1+\omega(\xi))\theta|-\lambda\omega(\xi)}$$

for $\zeta\in\sigma(\theta,t)$. Here $S(\theta)=\inf\{\langle x,\theta\rangle\,|\,x\in\operatorname{supp}\phi\}$. Estimating (13) by (12), (14), suitably large λ and small ε, we have

$$|\tilde{E}(\phi)|\leqq c_{\lambda,\varepsilon}\|\phi\|_{\lambda} \text{ for fixed } t \text{ and } \theta.$$

This implies that E is a fundamental solution of P in D_{ω}'. Further, the integral (13) is independent of (θ,t) for bounded $\theta\in\Gamma(P,N)$ and $t<-2c(\theta)$ because of (12), (14), Lemma 3.1 and the analyticity of the integrand on $\sigma(\theta,t)$.

To show that E has the support in $\Gamma^*(P,N)$, we take ϕ such that $\operatorname{supp}\phi\subset\{x\in\boldsymbol{R}^n:\langle x,\ \theta\rangle>0\}$, for given $\theta\in\Gamma(P,\ N)$. Then we have

(15) $$|\tilde{E}(\phi)|\leqq c_{\lambda,\varepsilon}\|\phi\|_{\lambda}|t|^{-m}e^{tS(\theta)+\varepsilon|t||\theta|}\int_{\sigma(\theta,t)}e^{t(S(\theta)-\varepsilon|\theta|)\omega(\xi)-\lambda\omega(\xi)}d\xi.$$

Since $S(\theta)$ is positive, we can choose ε so small that $S(\theta)-\varepsilon|\theta|>0$. Thus (15) can be estimated that for sufficiently large $\lambda>0$

$$|\tilde{E}(\phi)|\leqq c|t|^{-m}e^{t(S(\theta)-\varepsilon|\theta|)}$$

for some constant c independent of t. Letting $t\to-\infty$, we obtain $\tilde{E}(\phi)=0$, which means that supp E is in $\{x\in\boldsymbol{R}^n:\langle x,\theta\rangle>0\}$ for each $\theta\in\Gamma(P,N)$. This shows that supp E is in $\Gamma^*(P,N)$.

To show the final statement, let K be a closed convex cone with vertex at 0, which contains the support of the fundamental solution E. In view of Theorem 2.3, all the proper planes $\langle x,\ \theta\rangle=0$ of K, whose positive half-space contains K, are non-characteristic. Hence, the convex open set $K^*=\{\theta\in\boldsymbol{R}^n:\langle x,\theta\rangle>0$ for all $x\neq0$ in $K\}$ containing N is contained in $\{\theta: P_m(\theta)\neq0\}$, which gives that K^* is in $\Gamma(P,N)$. That is, K contains $\Gamma^*(P,N)$.

Combining Theorems 2.3 and 3.9 we have our main result:

THEOREM 3.10. *$P(D)$ is ω–hyperbolic with respect to N if and only if there is a fundamental solution in D_{ω}' whose support is contained in a proper cone of H.*

As a special case we have the following result for the distribution space.

COROLLARY 3.11. *Suppose $P(D)$ is a differential operator in $\mathbf{R}^n$ with constant coefficients. Then the following statements are equivalent:*

(a) There exists a unique fundamental solution in D' whose support is contained in a proper cone of H.

(b) There exists a constant c such that $P_m(N)\neq 0$ and, for $\tau<-c$ and $\xi\in\mathbf{R}^n$, $P(\xi+i\tau N)\neq 0$.

(c) There is a constant c such that $P_m(N)\neq 0$ and $P(\xi+i\tau N)\neq 0$ when $\tau<-c(1+log(1+|\xi|))$ and $\xi\in\mathbf{R}^n$.

When $\omega(\xi)=|\xi|^{1/d}$, $d>1$, our result is the d-hyperbolicity in [7]. We finally give an example of ω-hyperbolic operator.

EXAMPLE. Let $a_1,\cdots,a_n$ be n fixed real numbers such that $a_n\neq 0$. Let $P(D)$ be a differential operator defined by

$$P(D)=a_1\frac{\partial}{\partial x_1}+\cdots+a_{n-1}\frac{\partial}{\partial x_{n-1}}-a_n\frac{\partial^2}{\partial x_n^{\,2}}\cdot$$

Let $\omega(\xi)=|\xi|^{\frac{1}{2}}=(\xi_1^{\,2}+\cdots+\xi_n^{\,2})^{\frac{1}{4}}$. Then

$$P(\xi_1,\cdots,\xi_{n-1},\xi_n)=-i(a_1\xi_1+\cdots+a_{n-1}\xi_{n-1})+a_n\xi_n^{\,2}.$$

Taking $N=(0,\cdots,0,1)$, $P_m(N)=a_n\neq 0$ and $P(\xi+i\tau N)$ is given by $-i(a_1\xi_1+\cdots+a_{n-1}\xi_{n-1})+a_n(\xi_n+i\tau)^2$. Further, it can be shown that $P(\xi+i\tau N)=0$ if and only if $-(a_1\xi_1+\cdots+a_{n-1}\xi_{n-1})+2a_n\xi_n=0$, and $\xi_n^{\,2}=\tau^2$. We now choose a constant $C>0$ such that

$$4a_n^{\,2}C^4=n(1+a_1^{\,2}+\cdots+a_n^{\,2})$$

Then it can be seen that $P(\xi+i\tau N)\neq 0$ when $\tau<-C(1+|\xi|^{\frac{1}{2}})$ and $\xi\in\mathbf{R}^n$. That is, P is ω-hyperbolic with respect to N, $\omega(\xi)=|\xi|^{\frac{1}{2}}$. On the other hand, it can be easily seen that P is not hyperbolic with respect to N if a_n and some a_k are non-zero with $1\leq k\leq n-1$.

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Yonsei University
Seoul 120-749, Korea
and
Pusan University of Foreign Study
Pusan 608-060, Korea