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# EIGENVALUES OF THE LAPLACE OPERATOR ON A HERMITIAN VECTOR BUNDLE[1]

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## 0. Introduction

Let $M$ be a compact connected Kähler manifold of dimension $n$. The Kähler form is denoted by

$$\Phi=\sqrt{-1}\sum_{\mu,\nu=1}^{n} g_{\mu\bar{\nu}}dz^{\mu}\wedge d\bar{z}^{\nu}$$

and its cohomology class by $[\Phi]$.

Let $E$ be a hermitian holomorphic vector bundle over $M$ of rank $r$. We denote by $A^k(E)$ (resp. $A^{p,q}(E)$) the space of differential $k$-forms (resp. $(p,q)$-forms) on $M$ with values in $E$;

$$A^k(E)=\sum_{p+q=k} A^{p,q}(E).$$

Let $A^{\cdot}=\sum_k A^k$ be the (complex valued) exterior algebra of differential forms on $M$ and let $A^{\cdot}(E)=\sum_k A^k(E)$. The *Cauchy-Riemann operator*

$$d'' : A^0(E) \to A^{0,1}(E)$$

characterizes the holomorphic structure on $E$, since a local smooth section $s$ of $E$ is holomorphic if and only if $d''(s)=0$.

The *adjoint* of $d''$ is denoted by $\delta'' : A^{0,1}(E) \to A^0(E)$. The *Laplacian* we consider is

$$\Box :=\delta'' d'' : A^0(E)\to A^0(E).$$

This is a second order linear self-adjoint elliptic operator with nonnegative eigenvalues $\lambda_1\leq\lambda_2\leq\cdots\to\infty$. It is obvious that $\lambda_1=0$ if and only if $E$ admits a nontrivial holomorphic section.

Let $*$ be the conjugate linear isomorphism

$$* : A^0(E) \to A^0(E^*)$$

induced from the hermitian structure. This map extends naturally to a conjugate linear isomorphism

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$$* : A^\bullet(E) \to A^\bullet(E^*)$$

between the modules over the ring $A^\bullet$ of the (complex valued) differential forms on $M$. The isomorphism $*$ commutes with the Hodge star $*$ and the *Chern connection* $D$, the unique connection compatible with the holomorphic structure and the metric;

$$*(\xi^*) = (*\xi)^*, \ \ d_D(\xi^*) = (d_D\xi)^*$$

for $\xi \in A^\bullet(E)$, where

$$d_D : A^k(F) \to A^{k+1}(F)$$

is the natural extension of $D : A^0(F) \to A^1(F)$, for $F=E$ or $E^*$. It follows immediately from this observation that

$$d_D'(\xi^*) = (d_D''\xi)^*, \ \ d_D''(\xi^*) = (d_D'\xi)^*$$
$$\delta_D'(\xi^*) = (\delta_D''\xi)^*, \ \ \delta_D''(\xi^*) = (\delta_D'\xi)^*,$$

where $\delta_D = \delta_D' + \delta_D''$ denotes the adjoint of $d_D = d_D' + d_D''$. We have, therefore,

$$(0.1) \qquad \Delta'(\xi^*) = (\Delta''\xi)^*, \ \ \Delta''(\xi^*) = (\Delta'\xi)^*,$$

where $\Delta' = d_D'\delta_D' + \delta_D' d_D'$, $\Delta'' = d_D''\delta_D'' + \delta_D'' d_D''$. Hence we get

$$\mathrm{spec}(E, \Delta') = \mathrm{spec}(E^*, \Delta''),$$

as sets of eigenvalues with multiplicity. Note that $\Box = \Delta''$ on pure sections. This is the reason why we are interested in $\Box$-Laplacian instead of the full Laplacian $\Delta = \Delta' + \Box$.

Let $R = d_D \circ d_D : A^{p,q}(E) \to A^{p+1,q+1}(E)$ be the curvature tensor associated with the Chern connection $D$. Then the mean *curvature transformation* $K : A^0(E) \to A^0(E)$ is defined by [3]

$$K^i_j = \sum_{\mu,\nu} g^{\mu\bar{\nu}} R^i_{j\mu\bar{\nu}} \quad (1 \leq i, j \leq r).$$

$K$ is a hermitian symmetric endomorphism of $E$. Let $a(x)$ (resp. $b(x)$) be the minimum (resp. maximum) eigenvalue of the mean curvature transformation $K_x$ on the fiber $E_x$, $x \in M$. Then $a$ (resp. $b$) is a real valued continuous function on $M$ and its minimum (resp. maximum) value is denoted by $K_{min}$ (resp. $K_{max}$). We have, therefore,

$$(0.2) \qquad K_{min}\|s\|^2 \leq \langle\langle Ks, s \rangle\rangle \leq K_{max}\|s\|^2, \ \ s \in A^0(E),$$

where

$$\langle\langle s_1, s_2 \rangle\rangle = \int_M \langle s_1, s_2 \rangle(x)\,dx, \ \ \|s\|^2 = \langle\langle s, s \rangle\rangle.$$

The main theorem we prove is

THEOREM 0.3. *Let $E$ be a hermitian holomorphic vector bundle over* *a compact Kähler manifold M. Then*

$$K_{min} \le \lambda_k(E^*) - \lambda_k(E) \le K_{max}$$

*for* $k=1, 2, \cdots$.

This theorem is a generalization of the Kobayashi–Wu vanishing theorem ([4], [3]), since if $K_{max}<0$, then $\lambda_1(E)>0$ and hence $E$ admits no (nontrivial) holomorphic section.

In Section 2, we generalize the ordinary concept of the heat kernel and obtain its trace. In Section 3, we show that eigenvalues increase in both subbundles and quotient bundles.

## 1. Proof of Theorem 0.3 and remarks

Note that the mean curvature transformation appears in the Nakano identity [3]

$$\sqrt{-1}[\Lambda, R] = \Delta' - \Delta''$$

as operators on $A^k(E)$, where $\Lambda$ is the adjoint of the multiplication by the Kähler form $\Phi$. When $k=0$, we have

$$K = \Delta' - \Box.$$

Let $s_1, s_1, \cdots$ be an orthonormal basis of eigensections of $E$ with $\Box s_i = \lambda_i s_i$. Note that

$$\lambda_k(E) = \inf \langle\!\langle \Box s, s \rangle\!\rangle / \|s\|^2, \quad s \in A^0(E), \quad s \perp \{s_1, \cdots, s_{k-1}\}$$

$$\lambda_k(E^*) = \inf \langle\!\langle \Delta' s, s \rangle\!\rangle / \|s\|^2, \quad s \in A^0(E), \quad s \perp \{s_1, \cdots, s_{k-1}\}.$$

The second identity follows from the identity (0.1). Thus

$$\langle\!\langle \Box s, s \rangle\!\rangle = \langle\!\langle \Delta' s, s \rangle\!\rangle - \langle\!\langle K s, s \rangle\!\rangle \ge (\lambda_k(E^*) - K_{max}) \|s\|^2$$

for $s \perp \{s_1, \cdots, s_{k-1}\}$ and hence

$$\lambda_k(E) \ge \lambda_k(E^*) - K_{max}.$$

Similarly, for $s \perp \{s_1, \cdots, s_{k-1}\}$,

$$\langle\!\langle \Delta' s, s \rangle\!\rangle = \langle\!\langle \Box s, s \rangle\!\rangle + \langle\!\langle K s, s \rangle\!\rangle \ge (\lambda_k(E) + K_{min}) \|s\|^2$$

and hence

$$\lambda_k(E^*) \ge \lambda_k(E) + K_{min}.$$

This completes the proof.

When the mean curvature transformation is a constant multiple of the identity endomorphism, $E$ is called an *Einstein–Hermitian vector bundle*. In this case,

$$K = \rho_E 1_E,$$

where

$$\rho_E = +\frac{2\pi n}{r} c_1(E) \cup [\Phi]^{n-1}/[\Phi]^n,$$

$c_1(E)$ being the first Chern class of $E$. As a corollary, we have

COROLLARY 1.1. *If $E$ is an Einstein-Hermitan vector bundle, then*

$$\lambda_k(E^*) - \lambda_k(E) = \rho_E.$$

*In particular,*

$$\lambda_1(E) \geq -\rho_E.$$

REMARK 1.2. If $E$ is Einstein with $\rho_E = 0$, then it follows from the Kobayashi-Wu vanishing theorem [4] that every holomorphic section of $E$ is parallel and hence $\dim H^0(M, E) \leq r$. Thus $\lambda_{r+1} > 0$. Furthermore, if $E$ is (holomorphically) indecomposable, then $H^0(M, E) = 0$ and hence $\lambda_1 > 0$, unless $E$ is the trivial line bundle. This fact is also obtained from the vanishing theorem of stable bundles in algebraic geometry. Namely, a stable bundle of rank$\geq 2$ with $\rho \leq 0$ has no (non-trivial) holomorphic section [5].

## 2. Heat equation

Let $H_t (t \geq 0)$ be a section of the bundle $E \boxtimes E^*$ over $M \times M$. Thus for each $(x, y) \in M \times M$,

$$H_t(x, y) \in E_x \otimes E_y{}^* \simeq \mathrm{Hom}(E_y, E_x).$$

$H_t$ is called the *fundamental solution of the heat equation* (or the *heat kernel*) if

(a) $$\left(\frac{\partial}{\partial t} + \Box_x\right) H = 0$$

(b) $$\lim_{t \to 0} \int_M H_t(x, y) s(y) dy = s(x)$$

for any smooth section $s$ of $E$. This generalizes the ordinary concept of heat kernel ([1], [2]). In (b) the integrand $H_t(x, y)s(y)$ is a function on $M$ with values in $E_x$ and the integral is $E_x$-valued.

From (a) and (b), the general solution $s_t(x)$ of the heat equation with the initial condition $s_0(x) = s(x)$ is given by

$$s_t(x) = \int_M H_t(x, y) s(y) dy.$$

The heat kernel $H_t(x, y)$ is uniquely determined by the above properties (a) and (b). If $\{s_i\}$ is an orthonormal basis (with respect to the $L^2$–norm) of eigensections with eigenvalues $\lambda_i$, then

$$H_t(x, y)=\sum_i e^{-\lambda_i t} s_i(x)\otimes(s_i(y))^*,$$

where $(s_i(y))^*\in E_y^*$ is the dual of $s_i(y)\in E_y$ obtained from the conjugate linear isomorphism $E_y\to E_y^*$.

Let $Z_E(t)=\int_M \operatorname{tr} H_t(x, x)dx=\sum_i e^{-\lambda_i t}$, $t\geq 0$, and call it the *partition function* of $E$. Then $Z_E(t)$ determines all of the $\lambda_i$'s ([1], [2]). It is clear that

$$Z_E(0)=\dim H^0(M, E),$$

the maximal number of linearly independent holomorphic sections of $E$.

## 3. Second fundamental form

Let $E$ be a holomorphic subbundle of a hermitian holomorphic vetor bundle $F$. Then the connections of $E$ and $F$ are related by

$$D^F s=D^E s+I\!I s, \quad s\in A^0(E)$$

where $I\!I\in A^{1,0}(E, E^\perp)$ is the second fundamental form [3].

THEOREM 3.1. *The Laplacians of $E$ and $F$ are related by*

$$\Box^E s=p\circ\Box^F s, \quad s\in A^0(E)$$

*where $p: F\to E$ denotes the orthogonal projection.*

*Proof*. Locally, if $X_1, \cdots, X_n$ is a unitary tangent frame field on $M$ of type $(1, 0)$, then

$$\Box=-\sum_i D^2_{X_i\bar{X}_i},$$

where $D^2_{XY}=D_X D_Y-D_{D_X Y}$ [6]. It follows from this that

$$\Box^F s=\Box^E s-\sum_i I\!I_{X_i}\circ D^E_{\bar{X}_i} s, \quad s\in A^0(E).$$

Since $\Box^E s\in A^0(E)$ and $I\!I_{X_i} D^E_{\bar{X}_i}(s)\in A^0(E^\perp)$, we have $\Box^E s=p\circ\Box^F s$.

The following corollary is a direct consequence of the above theorem.

COROLLARY 3.2. *Spectrum increases in subbundles, i.e., $\lambda^E_k\geq\lambda^F_k$.*

In fact,

$$\operatorname{spec}(F, \Box^F)=\operatorname{spec}(E, \Box^E)\cup\operatorname{spec}(E^\perp, \Box^F|E^\perp)$$

as sets with multipicity. Thus we also have

$$\lambda^{E^\perp}_k\geq\lambda^F_k.$$

Since the quotient bundle $F/E$ is isometric to $E^{\perp}$,

COROLLARY 3.3. *Spectrum increases in quotient bundles.*

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