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A CONFORMAL CURVATURE TENSOR FIELD ON HERMITIAN MANIFOLDS

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In 1949 ([B]), S. Bochner has introduced "Bochner curvature tensor" on a kaehler manifold analogous to the Weyl conformal curvature tensor on a riemannian manifold. However, we have not known the exact meaning of his tensor yet. In this paper, we define a new tensor field on a hermitian manifold which is conformally invariant and study some properties of this new tensor field.

It is well known ([Be]) that the decomposition theorem of the curvature tensor R of a kaehler manifold;

(#) $$R=\frac{s}{2m^2}\omega\otimes\omega+\frac{1}{m}\omega\otimes\rho_0+\frac{1}{m}\rho_0\otimes\omega+B,$$

(##) $$B=\frac{\operatorname{tr} B}{m^2-1}Id\,|\,\Lambda_0{}^{1,1}M+B_0.$$

In this paper, we shall prove the above decomposition of the curvature tensor field of the hermitian connection on a hermitian manifold and B_0 is invariant under the conformal change of the hermitian metrics.

1. Notations and preliminaries

Let (M, g, J) be a hermitian manifold of complex dimension m. The complex structure J induces the splitting of the complexified tangent bundle $T_CM\simeq TM\otimes_R C$ into two complementary subbundles, conjugate to each other:

(1.1) $$T_CM\simeq T^{1,0}M\oplus T^{0,1}M,$$

where at each point x of M, the fiber $T_x{}^{1,0}M$ (resp. $T_x{}^{0,1}M$) is the eigenspace of J_x relative to eigenvalue $\sqrt{-1}$ (resp. $-\sqrt{-1}$). The elements of $T^{1,0}M$ (resp. $T^{0,1}M$) are the (complex) vectors of type

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(1, 0) (resp. of type (0, 1)).

The above splitting (1, 1) of the complexified tangent bundle $T_C M$ extends a splitting into types of the whole tensor bundle. In particular, we have

(1.2) $$\Lambda_C^r M = \sum_{p+q=r} \Lambda^p (T^{1,0}M)^* \otimes \Lambda^q (T^{0,1}M)^*,$$

where $\Lambda_C^r M$ is the bundle of the C-valued r-forms and $\Lambda^p(T^{1,0}M)^*$ (resp. $\Lambda^q(T^{0,1}M)^*$) denotes the bundle of the C-linear p-forms (resp. q-forms) on $T^{1,0}M$ (resp. $T^{0,1}M$).

Sections of the bundle $\Lambda_C^{p,q}M := \Lambda^p(T^{1,0}M)^* \otimes \Lambda^q(T^{0,1}M)^*$ are (complex) forms of type (p, q). In particular, if $p=q$, we denote by $\Lambda^{p,p}M$ the bundle of the real forms of type (p, p).

The fundamental 2-form ω of a hermitian manifold (M, g, J) is defined by

(1.3) $$\omega(X, Y) = g(JX, Y)$$

for any vector fields X and Y on M. By means of ω, we may define the linear operators L and its formal adjoint $\wedge$ acting on forms; for any form φ,

(1.4) $$L\varphi = \omega \wedge \varphi.$$

A form φ is said to be primitive if $\wedge\varphi = 0$. Then we have the following Lefschetz decomposition theorem on forms.

PROPOSITION 1.1. *Every r-form φ on a hermitian manifold of complex dimension m has a unique representation as a sum*

(1.5) $$\varphi = \sum_{p \geq (r-m)^+} L^p \varphi_p \qquad (r-m)^+ := max(0, r-m),$$

where φ_p are primitive $(r-2p)$-forms.

REMARK 1.2. For a 2-form φ of type (1, 1), the Lefschetz decomposition reduces to

(1.6) $$\varphi = \frac{\wedge\varphi}{m}\omega + \varphi_0,$$

where φ_0 is the primitive part of φ.

2. Decomposition of the curvature tensor fields

Let (M, g, J) be a hermitian manifold of complex dimension m. The hermitian connection ∇ of M is a unique affine connection such that g and J are parallel and the torsion tensor T satisfies;

$T(JX, Y)=T(X, JY)$ for any vector fields X and Y on M.

Let R be the curvature tensor of the hermitian connection ∇ of M;

(2.1) $$R(X, Y)=[\nabla_X, \nabla_Y]-\nabla_{[X,Y]}.$$

Then the curvature tensor R satisfies the following relations;

(2.2) $$R(X, Y)\circ J=J\circ R(X, Y),$$

(2.3) $$R(JX, JY)=R(X, Y)$$

for any vector fields X and Y on M.

We may consider that the curvature tensor R have the following properties;

(A) R is a section of the vector bundle $\Lambda^{1,1}M\otimes\Lambda^{1,1}M$.

(B) R is an endomorphism of the vector bundle $\Lambda^{1,1}M$.

Let $z^1, \cdots, z^m$ be a complex local coordinate system in M. The indices $i, j, k, \cdots$ run through $1, \cdots, m$ and the indices $A, B, C, \cdots$ run through $1, \cdots, m,\ 1^* \cdots, m^*$, and we also use the Einstein convention.

We set $Z_i=\partial/\partial z^i$, $Z_{i^*}=\partial/\partial \bar{z}^i$ and

(2.4) $$R(Z_C, Z_D)Z_B=R_{BCD}{}^A Z_A.$$

We also define

(2.5) $$R_{ABCD}=g(R(Z_C, Z_D)Z_B, Z_A),$$

so that

(2.6) $$R_{ABCD}=g_{AE}R_{BCD}{}^E.$$

The only nonvanishing components of the curvature tensor R of a hermitian connection ∇ are

$$R_{ij^*kl^*},\ R_{ij^*k^*l},\ R_{i^*jkl^*},\ R_{i^*jk^*l}.$$

Furthermore, we have

(2.7) $$R_{ij^*kl^*}=\frac{\partial^2 g_{ij^*}}{\partial z^k\partial\bar{z}^l}-g^{st^*}\frac{\partial g_{it^*}}{\partial z^k}\frac{\partial g_{sj^*}}{\partial\bar{z}^l}\quad \text{(see [G])}.$$

Since the hermitian connection ∇ has the torsion, there are three distinct Ricci tensors of ∇;

(2.8) $$R_{ij^*}=-g^{kl^*}R_{il^*kj^*},$$

(2.9) $$S_{ij^*}=-g^{kl^*}R_{ij^*kl^*},$$

(2.10) $$T_{ij^*}=-g^{kl^*}R_{kl^*ij^*}.$$

And we also have three distinct scalar curvatures of ∇ ;

(2.11) $$s' = 2g^{ij^*}R_{ij^*},$$

(2.12) $$s = 2g^{ij^*}S_{ij^*},$$

(2.13) $$t = 2g^{ij^*}T_{ij^*}.$$

Note that $s=t$.

By means of (A), R may be written locally as followings;

(2.14) $$R = R_{ij^*kl^*}dz^i \wedge d\bar{z}^j \otimes dz^k \wedge d\bar{z}^l.$$

Fixed k, l, $R_{ij^*kl^*}\, dz^i \wedge d\bar{z}_j$ is a real 2-form of type (1, 1). The Lefschetz decomposition (1.6) implies that

(2.15) $$R_{ij^*kl^*}dz^i \wedge d\bar{z}^j = -\frac{1}{m}T_{kl^*}g_{ij^*}dz^i \wedge d\bar{z}^j + {}^{0,1}R_{ij^*kl^*}dz^i \wedge d\bar{z}^j,$$

where ${}^{0,1}R_{ij^*kl^*}dz^i \wedge d\bar{z}^j$ is the primitive part of $R_{ij^*kl^*}\, dz^i \wedge d\bar{z}^j$, and similarly

(2.16) $$R_{ij^*kl^*}dz^k \wedge d\bar{z}^l = -\frac{1}{m}S_{ij^*}g_{kl^*}dz^k \wedge d\bar{z}^l + {}^{1,0}R_{ij^*kl^*}dz^k \wedge d\bar{z}^l,$$

where ${}^{1,0}R_{ij^*kl^*}\, dz^k \wedge d\bar{z}^l$ is the primitive part of $R_{ij^*kl^*}dz^k \wedge d\bar{z}^l$.

REMARK 2.1. The linear operator $\wedge$ can be written locally as followings;

(2.17) $$\wedge = \sqrt{-1}g^{kl^*}\, i(Z_k)i(Z_{l^*}).$$

Therefore we have

(2.18) $$\begin{aligned} 2R &= R_{ij^*kl^*}dz^i \wedge d\bar{z}^j \otimes dz^k \wedge d\bar{z}^l \\ &\quad + dz^i \wedge d\bar{z}^j \otimes R_{ij^*kl^*}dz^k \wedge d\bar{z}^l \\ &= \frac{\sqrt{-1}}{m}g_{ij^*}dz^i \wedge d\bar{z}^j \otimes \sqrt{-1}T_{kl^*}dz^k \wedge d\bar{z}^l \\ &\quad + dz^i \wedge d\bar{z}^j \otimes {}^{0,1}R_{ij^*kl^*}dz^k \wedge d\bar{z}^l \\ &\quad + \frac{\sqrt{-1}}{m}S_{ij^*}dz^i \wedge d\bar{z}^j \otimes \sqrt{-1}g_{kl^*}dz^k \wedge d\bar{z}^l \\ &\quad + {}^{1,0}R_{ij^*kl^*}dz^i \wedge d\bar{z}^j \otimes dz^k \wedge d\bar{z}^l. \end{aligned}$$

Since $\sqrt{-1}\ T_{kl^*}dz^k \wedge d\bar{z}^l$, $\sqrt{-1}\ S_{ij^*}dz^i \wedge d\bar{z}^j$, ${}^{0,1}R_{ij^*kl^*}dz^k \wedge d\bar{z}^l$ and ${}^{1,0}R_{ij^*kl^*}dz^i \wedge d\bar{z}^j$ are all real 2-forms of type (1, 1), we have, by (1.6),

$$(2.19)\qquad \sqrt{-1}\ T_{kl^*}dz^k\wedge d\bar{z}^l=\frac{s}{2m}\omega+\rho_{t,0},$$

$$(2.20)\qquad \sqrt{-1}\ S_{ij^*}dz^i\wedge d\bar{z}^j=\frac{s}{2m}\omega+\rho_{s,0},$$

$$(2.21)\qquad \begin{aligned} &{}^{0,1}R_{ij^*kl^*}dz^k\wedge d\bar{z}^l \\ &=\left(R_{ij^*kl^*}+\frac{1}{m}T_{kl^*}g_{ij^*}\right)dz^k\wedge d\bar{z}^l \\ &=\left(-\frac{1}{m}S_{ij^*}+\frac{s}{2m^2}g_{ij^*}\right)g_{kl^*}dz^k\wedge d\bar{z}^l \\ &\quad+{}^{0,1}B_{ij^*kl^*}dz^k\wedge d\bar{z}^l, \end{aligned}$$

$$(2.22)\qquad \begin{aligned} &{}^{1,0}R_{ij^*kl^*}dz^i\wedge d\bar{z}^j \\ &=\left(R_{ij^*kl^*}+\frac{1}{m}S_{ij^*}g_{kl^*}\right)dz^i\wedge d\bar{z}^j \\ &=\left(-\frac{1}{m}T_{kl^*}+\frac{s}{2m^2}g_{kl^*}\right)g_{ij^*}dz^i\wedge d\bar{z}^j \\ &\quad+{}^{1,0}B_{ij^*kl^*}dz^i\wedge d\bar{z}^j, \end{aligned}$$

where $\rho_{t,0}$, $\rho_{s,0}$, ${}^{0,1}B_{ij^*kl^*}\ dz^k\wedge d\bar{z}^l$ and ${}^{1,0}B_{ij^*kl^*}\ dz^i\wedge d\bar{z}^j$ are the primitive parts, respectively. Substituting (2.19)～(2.22) to (2.18), we have

$$(2.23)\qquad \begin{aligned} 2R=&\frac{1}{m}\omega\otimes\left(\frac{s}{2m}\omega+\rho_{t,0}\right) \\ &+\frac{\sqrt{-1}}{m}\left(S_{ij^*}-\frac{s}{2m}g_{ij^*}\right)dz^i\wedge d\bar{z}^j\otimes\sqrt{-1}\ g_{kl^*}dz^k\wedge d\bar{z}^l \\ &+\frac{1}{m}\left(\frac{s}{2m}\omega+\rho_{s,0}\right)\otimes\omega \\ &+\frac{\sqrt{-1}}{m}g_{ij^*}dz^i\wedge d\bar{z}^j\otimes\sqrt{-1}\left(T_{kl^*}-\frac{s}{2m}g_{kl^*}\right)dz^k\wedge d\bar{z}^l \\ &+{}^{0,1}B_{ij^*kl^*}dz^i\wedge d\bar{z}^j\otimes dz^k\wedge d\bar{z}^l \\ &+{}^{1,0}B_{ij^*kl^*}dz^i\wedge d\bar{z}^j\otimes dz^k\wedge d\bar{z}^l \\ =&\frac{s}{m^2}\omega\otimes\omega+\frac{2}{m}\omega\otimes\rho_{t,0}+\frac{2}{m}\rho_{s,0}\otimes\omega+{}^{0,1}B+{}^{1,0}B. \end{aligned}$$

However we can easily check that ${}^{0,1}B={}^{1,0}B$, and so we can set $B:={}^{0,1}B={}^{1,0}B$. Thus we have the decomposition of R ;

$$(2.24)\qquad R=\frac{s}{2m^2}\omega\otimes\omega+\frac{1}{m}\omega\otimes\rho_{t,0}+\frac{1}{m}\rho_{s,0}\otimes\omega+B,$$

where B is a section of the subbundle $\Lambda_0^{1,1}M\otimes\Lambda_0^{1,1}M$ and $\Lambda_0^{1,1}M$ is the vector bundle of real primitive 2-forms of type (1, 1).

By means of complex local coordinates, B can be written as followings;

$$B_{ij^*kl^*}=R_{ij^*kl^*}+\frac{1}{m}(g_{ij^*}T_{kl^*}+S_{ij^*}g_{kl^*}) - \frac{s}{2m^2}g_{ij^*}g_{kl^*}. \tag{2.25}$$

By means of (B), we have

$$\mathrm{tr}\, R=R^{ij^*}{}_{ij^*}=\frac{s'}{2}, \quad \mathrm{tr}\, R:=\text{the trace of } R. \tag{2.26}$$

Now, by means of the Lefschetz decomposition (1.6), the bundle $\Lambda^{1,1}M$ decomposes as followings;

$$\Lambda^{1,1}M=\Lambda_0{}^{1,1}M\oplus \boldsymbol{R}\cdot\omega \tag{2.27}$$

where $\boldsymbol{R}\cdot\omega$ denotes the trivial (real) line subbundle of $\Lambda^{1,1}M$ generated by the fundamental 2-form ω. Considered as an endomorphism of $\Lambda^{1,1}M$, we have $B(\boldsymbol{R}\cdot\omega)=0$. Therefore the trace of B as an endomorphism of $\Lambda^{1,1}M$ is equal to one as an endomorphism of $\Lambda_0{}^{1,1}M$. Thus (2.25) implies that

$$\mathrm{tr}\ B=B^{ij^*}{}_{ij^*}=\frac{ms'-s}{2m}. \tag{2.28}$$

Similarly considered as an endomorphism of $\Lambda_0{}^{1,1}M$, the tensor B decomposes as followings;

$$\begin{aligned} B&=\frac{\mathrm{tr}\ B}{m^2-1}Id\,|\,\Lambda_0{}^{1,1}M+B_0 \\ &=\frac{ms'-s}{2m(m^2-1)}Id\,|\,\Lambda_0{}^{1,1}M+B_0, \end{aligned} \tag{2.29}$$

where the trace of B_0 is null.

By means of complex local coordinates, B_0 is written as followings;

$$\begin{aligned} B_{0,ij^*kl^*}&=B_{ij^*kl^*}+\frac{ms'-s}{2m(m^2-1)}\left(g_{il^*}g_{kj^*}-\frac{1}{m}g_{ij^*}g_{kl^*}\right) \\ &=R_{ij^*kl^*}+\frac{1}{m}(g_{ij^*}T_{kl^*}+S_{ij^*}g_{kl^*}) \\ &\quad -\frac{ms'+(m^2-2)s}{2m^2(m^2-1)}g_{ij^*}g_{kl^*} \\ &\quad +\frac{ms'-s}{2m(m^2-1)}g_{il^*}g_{kj^*}. \end{aligned} \tag{2.30}$$

In particular, if M is kaehler, then the Ricci tensor R_{ij^*}, S_{ij^*}, T_{ij^*} are equal to each other and so necessarily $s'=s$. Therefore the decomposition (2.24) coincides with (#) and B_0 is written locally as followings;

$$B_{0,ij^*kl^*}=R_{ij^*kl^*}+\frac{1}{m}(g_{ij^*}R_{kl^*}+R_{ij^*}g_{kl^*}) -\frac{(m+2)s}{2m^2(m+1)}g_{ij^*}g_{kl^*}+\frac{s}{2m(m+1)}g_{il^*}g_{kj^*}. \tag{2.31}$$

3. Conformal invariance of B_0

Let (M, g, J) be a hermitian manifold with hermitian metric g, and $\tilde{g}=e^{\sigma}g$ be a conformal change of g where σ is any smooth real-valued function on M. Then we have

$$\tilde{R}_{ij^*kl^*}=e^{\sigma}\left(R_{ij^*kl^*}+\frac{\partial^2\sigma}{\partial z^k\partial\bar{z}^l}g_{ij^*}\right), \tag{3.1}$$

$$\tilde{R}_{ij^*}=R_{ij^*}-\frac{\partial^2\sigma}{\partial z^i\partial\bar{z}^j}, \tag{3.2}$$

$$\tilde{S}_{ij^*}=S_{ij^*}-\frac{\partial^2\sigma}{\partial z^s\partial\bar{z}^t}g^{st^*}g_{ij^*}, \tag{3.3}$$

$$\tilde{T}_{ij^*}=T_{ij^*}-m\frac{\partial^2\sigma}{\partial z^i\partial\bar{z}^j}, \tag{3.4}$$

$$\tilde{s}'=e^{-\sigma}\left(s'-2\frac{\partial^2\sigma}{\partial z^s\partial\bar{z}^t}g^{st^*}\right), \tag{3.5}$$

$$\tilde{s}=e^{-\sigma}\left(s-2m\frac{\partial^2\sigma}{\partial z^s\partial\bar{z}^t}g^{st^*}\right). \tag{3.6}$$

By means of (3.1)~(3.6), we have

$$\tilde{B}_{0,ij^*kl^*}=e^{\sigma}B_{0,ij^*kl^*}, \tag{3.7}$$

so that

$$\tilde{B}_{0,jkl^*}{}^{i}=B_{0,jkl^*}{}^{i}. \tag{3.8}$$

Summing up, we have

THEOREM A. *Let (M, g, J) be a hermitian manifold of complex dimension m $(m\geq 2)$. Then B_0 is conformally invariant.*

If (M, g, J) is kaehler and B_0 vanishes everywhere, then we have

$$R_{ij^*kl^*}=-\frac{1}{m}(g_{ij^*}R_{kl^*}+R_{ij^*}g_{kl^*}) \tag{3.9}$$

$$+\frac{(m+2)s}{2m^2(m+1)}g_{ij^*}g_{kl^*}-\frac{s}{2m(m+1)}g_{il^*}g_{kj^*}.$$

Transvecting g^{kj^*} to (3. 9), we have

$$-R_{il^*}=-\frac{2}{m}R_{il^*}+\frac{(m+2)s}{2m^2(m+1)}g_{il^*}-\frac{s}{2(m+1)}g_{il^*}.$$

Thus, if $m\geq 3$, we have

$$R_{il^*}=\frac{s}{2m}g_{il^*}. \tag{3.10}$$

Since (3. 10) represents the 1st Chern class, s is constant. Substituting (3. 10) to (3. 9), we have

$$R_{ij^*kl^*}=-\frac{s}{2m(m+1)}(g_{ij^*}g_{kl^*}+g_{il^*}g_{kj^*}). \tag{3.11}$$

Summing up, we have

THEOREM B. *Let (M, g, J) be a kaehler manifold of complex dimension m $(m\geq 3)$. Then M is of constant holomorphic sectional curvature if and only if B_0 vanishes everywhere.*

COMMENTS. The tensor field B_0 is conformally invariant even if $m=2$. However, when $m=2$, we can not have an assertion of Theorem B.

If M is a kaehler–Einstein manifold and $m=2$, then Theorem B holds good.

4. Real version of the tensor field B_0

Let (M, g, J) be a kaehler manifold of complex dimension m. The indices $\alpha, \beta, \gamma, \cdots,$ run through $1, \cdots, 2m$. We set the metric $g:=(g_{\alpha\beta})$ and $J:=(J_{\alpha\beta})$ in real coordinates, which satisfy the following relations;

$$g_{\gamma\varepsilon}J_\alpha{}^\gamma J_\beta{}^\varepsilon=g_{\alpha\beta},\quad J_\gamma{}^\alpha J_\beta{}^\gamma=-\delta_\beta{}^\alpha. \tag{4.1}$$

And we set $J_{\alpha\beta}=g_{\alpha\gamma}J_\beta{}^\gamma$, so that $J_{\alpha\beta}=-J_{\beta\alpha}$.

Let $R=(R_{\beta\gamma\varepsilon}{}^\alpha)$ be the riemannian curvature tensor and $\rho=(R_{\alpha\beta})=(R_{\alpha\gamma\beta}{}^\gamma)$ the Ricci tensor, s the scalar curvature. Then it is well known that

$$R_{\alpha\beta\lambda\mu}J_\gamma{}^\lambda J_\varepsilon{}^\mu=R_{\alpha\beta\gamma\varepsilon},\quad R_{\alpha\beta}J_\gamma{}^\alpha J_\varepsilon{}^\beta=R_{\gamma\varepsilon}. \tag{4.2}$$

We set $S_{\alpha\beta}=R_{\alpha\gamma}J_{\beta}{}^{\gamma}$, so that $S_{\alpha\beta}=-S_{\beta\alpha}$.

The real version of conformal invariant curvature tensor field B_0 is given by

(4.3)
$$\begin{aligned} B_{0,\alpha\beta\gamma\varepsilon} &= R_{\alpha\beta\gamma\varepsilon}+\frac{1}{2m}(g_{\alpha\gamma}R_{\beta\varepsilon}-g_{\alpha\varepsilon}R_{\beta\gamma}+R_{\alpha\gamma}g_{\beta\varepsilon}-R_{\alpha\varepsilon}g_{\beta\gamma} \\ &\quad -J_{\alpha\gamma}S_{\beta\varepsilon}+J_{\alpha\varepsilon}S_{\beta\gamma}-S_{\alpha\gamma}J_{\beta\varepsilon}+S_{\alpha\varepsilon}J_{\beta\gamma}-2J_{\alpha\beta}S_{\gamma\varepsilon}-2S_{\alpha\beta}J_{\gamma\varepsilon}) \\ &\quad +\frac{(m+2)s}{4m^2(m+1)}(J_{\alpha\gamma}J_{\beta\varepsilon}-J_{\alpha\varepsilon}J_{\beta\gamma}+2J_{\alpha\beta}J_{\gamma\varepsilon}) \\ &\quad -\frac{(3m+2)s}{4m^2(m+1)}(g_{\alpha\gamma}g_{\beta\varepsilon}-g_{\alpha\varepsilon}g_{\beta\gamma}). \end{aligned}$$

Let b_0 denote the Ricci contraction of B_0, i.e.,

(4.4) $$b_{0,\alpha\beta}=B_{0,\alpha\gamma\beta}{}^{\gamma}.$$

Transvecting $g^{\alpha\gamma}$ to (4.3), we have

(4.5) $$b_{0,\alpha\beta}=\frac{2(m-2)}{m}\left(R_{\alpha\beta}-\frac{s}{2m}g_{\alpha\beta}\right).$$

LEMMA 4.1. *If $m\geq 3$, then the Ricci contraction b_0 of B_0 vanishes everywhere if and only if M is Einstein.*

Moreover we have

(4.6) $$\nabla_\gamma b_{0,\alpha\beta}=\frac{2(m-2)}{m}\left(\nabla_\gamma R_{\alpha\beta}-\frac{1}{2m}g_{\alpha\beta}\nabla_\gamma s\right).$$

If $m\geq 3$ and b_0 is parallel, then we have

(4.7) $$\nabla_\gamma R_{\alpha\beta}=\frac{1}{2m}g_{\alpha\beta}\nabla_\gamma s.$$

Transvecting $g^{\beta\gamma}$ to (4.7), we have

(4.8) $$\nabla_\gamma R_\alpha{}^\gamma=\frac{1}{2m}\nabla_\alpha s.$$

On the other hand, by means of the second Bianchi identity, we have

(4.9) $$\nabla_\gamma R_\alpha{}^\gamma=\frac{1}{2}\nabla_\alpha s.$$

(4.8) together with (4.9) means

(4.10) $$\nabla_\alpha s=0$$

and so that the Ricci tensor is parallel.

LEMMA 4.2. *If $m \geq 3$, then the Ricci contraction b_0 of B_0 is parallel if and only if the Ricci tensor is parallel.*

The following classification theorem ([S]) is well known; the only (simply connected) complete Einstein complex hypersurfaces M in C^{m+1} (resp. $D^{m+1}(C)$), $m \geq 2$, are C^m (resp. $D^m(C)$). The only complete Einstein complex hypersurfaces M in $P^{m+1}(C)$, $m \geq 2$, are $P^m(C)$ or complex quadric $Q^m(C)$. T. Takahashi ([T]) showed that the condition that M is Einstein can be relaxed to the condition that the Ricci tensor ρ of M is parallel.

Therefore we have the following corollaries.

COROLLARY 4.3. *If $m \geq 2$, then there are no (simply connected) complete Einstein complex hypersurfaces with nonvanishing B_0 in* C^{m+1} and $D^{m+1}(C)$.

COROLLARY 4.4. *If $m \geq 2$, then the complex quadric $Q^m(C)$ is the only complete Einstein complex hypersurface with nonvanishing B_0 in $P^{m+1}(C)$.*

COROLLARY 4.5. *If $m \geq 3$, then there are no (simply connected) complete complex hypersurfaces with nonvanishing B_0 in C^{m+1} and D^{m+1} such that b_0 is parallel.*

COROLLARY 4.6. *If $m \geq 3$, then the complex quadric $Q^m(C)$ is the only complex hypersurface with nonvanishing B_0 in $P^{m+1}(C)$ such that b_0 is parallel.*

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References

[Be]. A.L. Besse, *Einstein manifolds*, Ergebnisse der Math. 3 Folge 10, Springer-Verlag, Berlin Heidelberg New York London Paris Tokyo, 1987.

[B]. S. Bochner, *Curvature and Betti numbers, II*, Ann. of Math. **50**(1949), 77–93.

[K]. S. Kobayashi, *Hypersurfaces of complex projective space with constant scalar*

curvature, J. Differential Geometry, **1**(1967), 369–370.
[K-N]. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. II, Interscience, Wiley, New York, 1969.
[G]. S. I. Goldberg, *Curvature and homology*, Academic Press, New York, 1962.
[S]. B. Smyth, *Differential geometry of complex hypersurfaces*, Ann. of Math. **85**(1967), 246–266.
[T]. T. Takahashi, *Hypersurface with parallel Ricci tensor in a space of constant holomorphic sectional curvature*, J. Math. Soc. Japan **19**(1967), 199–204.

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