J. Korean Math. Soc. 27(1990), No. 1, pp. 1-5

A NOTE ON IDEALS WHICH ARE MAXIMAL AMONG NONVALUATION IDEALS

Byung Gyun Kang*

In this paper R will be an integral domain. A Noetherian ring with unique maximal ideal is called a local ring. An ideal of R is called a valuation ideal if it is the contraction of an ideal of some valuation overring of R. It is known that every primary ideal of a Noetherian domain R is a valuation ideal if and only if R is a Dedekind domain. From this fact we come to be interested in the ideals which are maximal among nonvaluation ideals. One might guess that such an ideal has to be a primary ideal, but this is false. We will show thas such an ideal I in a Noetherian domain R is a primary ideal if and only if its radical $\sqrt{I}$ is a maximal ideal. In the case that R is a two dimensional regular local ring, we will show that I is a primary ideal. Note that $\sqrt{I}$ is not always a prime ideal. But it will turn out that $\sqrt{I}$ is a prime ideal if R is a local domain. This will be used to prove that in a two dimensional regular local ring, I is always a primary ideal. For undefined terms and general information, the reader is referred to [2].

Lemma 1. *Let R be a commutative ring such that the set $Z(R)$ of zero divisors is a union of finite number of prime ideals. Then any regular ideal of R is generated by regular elements.*

Proof. This follows from [1, Lemma B]

Lemma 2. *Let R be a local domain and I an ideal of R. If I is maximal among nonvaluation ideals of R, then $\sqrt{I}$ is a prime ideal.*

Proof. Let M be the maximal ideal of R. If $\sqrt{I}=M$, then there is nothing to prove. So let us assume that $\sqrt{I}\subsetneq M$. Choose $a\in M\setminus$

Received March 24, 1989.
*This research was supported by the POSTECH research fund.

$\sqrt{I}$. Then for each $k\geq 1$, $I\subsetneq I+(a^k)$. Now by passing to R/I and using Krull's intersection theorem [3, Theorem 142], we deduce that $I=\cap_{k=1}^{\infty}(I+(a^k))$. Put $I+(a^k)=I_k$. Now $I=\cap_{k=1}^{\infty}I_k$, $I_1\supseteq I_2\cdots\supseteq I_k\supseteq I_{k+1}\cdots$, and each I_k is a valuation ideal. To prove that $\sqrt{I}$ is a prime ideal, suppose that $xy\in I^2$ for $x, y\in R$. Then $xy\in(I_k)^2$ for each k. So either x or y is in I_k since I_k is a valuation ideal [2, Lemma 24. 4]. Hence at least one of x and y is contained in infinitely many I_k's, which implies that either x or y is contained in $\cap_{k=1}^{\infty}I_k=I$. Now suppose $xy\in\sqrt{I}$ for $x, y\in R$. Then $(xy)^n\in I$ for some $n>0$. So $x^{2n}y^{2n}=(xy)^{2n}\in I^2$. From the previous argument, either x^{2n} of y^{2n} is contained in I. From this, we conclude that x or $y\in\sqrt{I}$ and hence $\sqrt{I}$ is a prime ideal.

LEMMA 3. *Let R be an integral domain and I an ideal which is maximal among nonvalution ideals. If P is a prime ideal containing I, then R/P is a valuation ring.*

Proof. Let $\bar{x}, \bar{y}$ be two nonzero elements of $\bar{R}=R/P$, so that $x\notin P$, $y\notin P$. Since $I\subseteq P\subset P+(xy)$, we have that $P+(xy)$ is a valuation ideal of R. For some valuation overring V *of* R, $(xy)+P=((xy)+P)V\cap R$.

Then either

$$[x^2V\subseteq((xy)+P)V \text{ or } y^2V\subseteq((xy)+P)V]$$

or

$$[x^2V\supseteq((xy)+P))V \text{ and } y^2V\supseteq((xy)+P))V]$$

Case I. $x^2V\subseteq((xy)+P))V\Rightarrow x^2\in((xy)+P))V\cap R=(xy)+p\Rightarrow x^2=rxy+p$ for $r\in R$ and $p\in P\Rightarrow x(x-ry)\in P\Rightarrow x-ry\in P$ since $x\notin P$ $\Rightarrow x\in(y)+P\Rightarrow(x)+P\subseteq(y)+P\Rightarrow(\bar{x})\subseteq(\bar{y})$.

Case II. $((xy)+P)V\subseteq x^2V$ and $((xy)+P)V\subseteq y^2V\Rightarrow xy=x^2v$, $xy=y^2v'$ for some $v, v'\in V\Rightarrow x^2y^2=x^2y^2vv'\Rightarrow vv'=1\Rightarrow x^2=xyv'$ from $xy=x^2v\Rightarrow x^2\in((xy)+P)V\cap R$. This reduces to case I. Thus either $(\bar{x})\subseteq(\bar{y})$ or $(\bar{y})\subseteq(\bar{x})$. Hence R/P is a valuation ring.

COROLLARY 4. *Let R be a local domain and I an ideal maximal among nonvaluation ideals. Then $R/\sqrt{I}$ is a principal ideal domain.*

Proof. This follows from Lemma 2 and Lemma 3.

Let R be a Noetherian domain and I an ideal maximal among nonvaluation ideals. Let D be a Dedekind domain which is not a DVR. The set of nonvaluation ideals of D is not empty. Let us choose an ideal I which is maximal among nonvaluation ideals. If I is a primary, then $I=I_P \cap R$, where $P=\sqrt{I}$, and hence I is a valuation ideal since D_P is a DVR. This contradicts our choice of I. So I need not be a primary ideal. In the next theorem, we give a necessary and sufficient condition for I to be a primary ideal.

THEOREM 5. *Let D be a Noetherian domain and I an ideal maximal among nonvaluation ideals. Then I is a primary ideal if and only if $\sqrt{I}$ is a maximal ideal.*

Proof. ($\Rightarrow$) Suppose that I is a primary ideal. Let $\sqrt{I}=P$. We want to show that P is a maximal ideal. If not, there exists a maximal ideal M such that $P \subset M$. In $R=D/I, Z(R)=P/I$. Let $\bar{x}$ and $\bar{y}$ be regular elements of R, so $x, y, xy \notin P$. Then $I+(xy) \supseteq I$ so $I+(xy)$ is a valuation ideal of D. Then for some valuation overring V of R, $I+(xy)=(I+(xy))V \cap D$. As in the proof of Lemma 3, we deduce that either $x^2V \subseteq ((xy)+I)V$ or $y^2V \subseteq ((xy)+I)V$. We may assume that $x^2V \subseteq ((xy)+I)V$. We can find $r \in R$ such that $x(x-ry) \in I$ as we did in the case I of the proof of Lemma 3. Since I is a primary ideal and $x \notin P=\sqrt{I}$, so $(\bar{x}) \subseteq (\bar{y})$. Thus in D/I, the regular principal ideals are totally ordered. It is easy to see that every element of $(M \backslash P)/I$ is a regular element of D/I. So M/I is a regular ideal and it is generated by regular elements by Lemma 1. Since D/I is Noetherian, M/I is finitely generated and hence M/I is a principal ideal. By the Krull's principal ideal theorem, M/I is a minimal prime ideal of D/I. So $M/I=P/I$ and $M=P$, which contradicts our assumption that $P \subset M$. Therefore we conclude that P is a maximal ideal.

($\Leftarrow$) is obvious.

Let $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n$ be a chain of prime ideals of an integral domain R. Then there always exists a valuation overring V of R such that $P_iV \cap D=P_i$ for each $i=1, \cdots, n$. This fact is crucial in proving the next result.

THEOREM 6. *Let (R, M) be a two dimensional regular local domain and I an ideal maximal among nonvaluation ideals of R. Then $\sqrt{I}$ is the maximal ideal of R.*

Proof. If $P=\sqrt{I}$ is not the maximal ideal, then P is a minimal prime ideal of R. Since R is a UFD, there exists an $a\in R$ such that $P=(a)$. By corollary 4, R/P is a PID. So $M=P+(b)$ for some $b\in R$ and $M=(a, b)$. It is easy to see that $A\equiv\{J$ is an ideal of $R \mid (a^2)\subseteq J \subseteq (a)\}=\{(a^2, ab^k)\}_{k=0}^{\infty}$. We claim that $I\in A$. We have to show that $P^2\subseteq I$. For otherwise, $P^2\not\subseteq I$ and $P^2\not\subseteq I+(b^n)$ for some n by Krull's intersection theorem. Let $J=(I+(b^n))\cap P$. Then $Pb^n\in J$. Since $I\subseteq J \subseteq P$, we have that $\sqrt{I}\subseteq\sqrt{J}\subseteq P$. Thus $\sqrt{J}\subseteq M$. Choose $z\in M\setminus \sqrt{J}$. Then following the same argument as in the proof of Lemma 2, we can showt that $J=\cap_{k=1}^{\infty}(J+(z^k))$. Put $J+(z^k)=J_k$. Then each J_k is a valuation ideal since J_k properly contains I. Now let $x=a, y=b^n$. Now $xy\in J$, which implies hat $xy\in J_k$ for each k. For eacth k, either x^2 or y^2 belongs to J_k since J_k is a valuation ideal [2, Lemma 24.4], and hence either x^2 or y^2 belongs to infinitely many J_k. So $x^2\in J=\cap_{k=1}^{\infty} J_k$ or $y^2\in J$, i.e., $a^2\in J$ or $b^{2n}\in J$. This contradicts that $P^2\not\subseteq I$ and $b\notin P$. Thus we have that $P^2\subseteq I$. Now $I\in A$, so that $I=(a^2, ab^n)$ for some $n\geq 0$. We can choose a valuation domain V such that $PV\cap R=P$ and $MV\cap R=M$. Obviously $IV\cap R\in A$, so $IV\cap R=(a^2, ab^k)$ for some k. But $k\leq n$ since $I\subseteq IV\cap R$. We will show that $k=n$, so that $I=IV\cap R$. Suppose $k<n$. Then

$$\begin{aligned} ab^k\in IV=(a^2, ab^n)V &\Rightarrow b^k\in (a, b^n)V \\ &\Rightarrow b^k(1-b^{n-k}v)\in aV \text{ for some } v\in V \\ &\Rightarrow b^k\in aV \text{ since } 1-b^{n-k}v\,(n-k>0) \text{ is a unit of } V \\ &\quad \text{(note that } b \text{ is a nonunit of } V \text{ since } b\in M \text{ and } MV\neq V) \\ &\Rightarrow b^k\in aV\cap D=P \\ &\Rightarrow b\in P, \end{aligned}$$

which contradicts that $P\neq M$. Thus $k=n$, so $I=IV\cap R$ is a valuation ideal. But this contradicts that I is not a valuation ideal. Therefore $\sqrt{I}$ is the maximal ideal of R.

References

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Pohang Institute of Science and Technology
Pohang 790-330, Korea