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A Note on the Weak* Radon Nikodym Property

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ABSTRACT. In this paper, we introduce the notion of the compact range property and weak* Radon Nikodym property. We prove that the compact range property, weak Radon Nikodym property and weak* Radon Nikodym property in dual Banach space are all equivalent. Other related results and discussed.

1. Introduction and Preliminaries

In this paper, we introduce the notion of the compact range property and weak* Radon Nikodym property and investigate some properties of compact range property and weak* Radon Nikodym property. We prove that the dual space X^* has compact range property if and only if the dual space X^* has weak* Radon Nikodym property, if and only if the dual space X^* has weak Radon Nikodym property.

Throughout this paper, $([01], \sum, \mu)$ denotes a Lebesgue measure space and $(\Omega, \sum, \mu)$ denotes a finite measure space.

The following definitions are found in [**3**]

DEFINITION 1.1: Let $(\Omega, \sum, \mu)$ be a finite measure space, and let X be a Banach space, and let $T : L_1(\mu) \to X$ be a bounded linear operator. A function $\phi : \Omega \to X$ is called a Pettis density for T if it is Pettis integrable, scalarly bounded and $\langle T(g), x^* \rangle = \inf g \langle x^*, \phi \rangle \, d\mu$ for all x^* in X^* and all g in $L_1(\mu)$. Also, a Pettis density which is strongly measurable is called a Bochner density.

DEFINITION 1.2: A Banach space X has the weak Radon Nikodym property (WRNP) if each bounded linear operator $T : L_1([01], \sum, \mu) \to X$ has a Pettis density.

DEFINITION 1.3: A Banach space X has the weak star Radon Nikodym property (W*RNP) if each bounded linear operator $T : L_1([01], \sum, \mu) \to X$ has a Pettis density ϕ valued in X^{**}.

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DEFINITION 1.4: Let X be Banach space and $T : L_1([01], \sum, \mu) \to X$ be a bounded linear operator. If T take weakly compact sets into norm compact sets, then T is called to be a Dunford Pettis operator.

DEFINITION 1.5: A Banach space X has the compact range property (CRP) if each bounded linear operator $T : L_1([01], \sum, \mu) \to X$ is a Dunford Pettis operator.

The above definition is due to Dunford and Pettis [**3**]. Observe that a compact operator must be a Dunford Pettis operator. Clearly the two notions coincide when X is reflexive. In [**3**], Dunford and Pettis showed that all reflexives and separable dual Banach spaces have the compact range property. Nowadays, by Talagrand, it is proved that the space with Radon Nikodym property has the compact range property. Recently J.J Uhl proved many deep results concerning the Banach space which satisfy the compact range property.

A finite measure space $(\Omega, \sum, \mu)$ is called to be perfect if for each measurable function $f : \Omega \to R$ and each set $F \subset R$ for which $f^{-1}(F)$ in $\sum$, there is Borel set $G \subset F$ with $\mu f^{-1}(G) = \mu f^{-1}(F)$.

PROPOSITION 1.6. *A finite Radon measure is perfect.*

PROOF: Let $F \subset R$ for which $f^{-1}(F) \in \sum$. Since μ is Radon measure. For each n, there is a compact set $k_n \subset f^{-1}(F)$ with $\mu(k_n) \geq \mu f^{-1}(F) - 2^{-n}$, on which f is continuous. Take $G = \cap_n f(k_n)$. Then G is a Borel set and $\mu f^{-1}(G) = \mu f^{-1}(F)$. Hence μ is perfect.

COROLLARY 1.7. *$([01], \sum, \mu)$ is perfect.*

The following propositions are found in [**1**]

PROPOSITION 1.8: Let $(\Omega, \sum, \mu)$ be a perfect and let X be a Banach space. Then a bounded linear operator $T : L_1(\mu) \to X$ which has a Pettis density valued in X^{**} is a Dunford Pettis operator.

By the above proposition 1.8 and the corollary 1.7, we know that the weak* Radon Nikodym property and weak Radon Nikodym property imply the compact range property.

PROPOSITION 1.9: $L_1([01], \sum, \mu)$ and c_0 fail the compact range property.

The following corollary can be easily abstained from Proposition 1.9.

COROLLARY 1.10. *A Banach space with W*RNP (resp. WEAK, RNP) never contains* c_0 *or* $L_1([01], \sum, \mu)$

2. Main results

The following theorems are well-known.

THEOREM 2.1[In [4], K. Musial]. *A dual Banach space* X^* *has the weak Radon Nikodym theorem if and only if* X *does not contain copy of* ℓ_1.

THEOREM 2.2[Pelczynski's Theorem]. *The following statements about a Banach space* X *are equivalent.*

(a) *The space* X *contains no copy of* ℓ_1.
(b) *Every bounded linear operator from* $L_1([01], \sum, \mu)$ *into* X^* *is Dunford Pettis operator.*
(c) *The dual space* X^* *contains no copy of* $L_1([01], \sum, \mu)$.

The following Theroem 2.3 can be abtained from Theorem 2.1 and Theorem 2.2.

THEOREM 2.3. *Let* X *be a Banach space. Then the dual space* X^* *has the compact range property if and only if the dual space* X^* *has the weak Radon Nikodym property.*

PROOF: Let X^* have the compact range property. By Theorem 2.2, then X contains no copy of ℓ_1. Also, we abtain from Theorem 2.1. that X^* has the weak Radon Nikodym property. Conversely let X^* have the weak Radon Nikodym property. By Proposition 1.8, we abtain that X^* has the compact range property.

THEOREM 2.4. *Let* X *be a Banach space. Then a dual Banach space* X^* *has the weak* Radon Nikodym, Property if and only if a dual Banach space* X^* *has weak Radon Nikodym property.*

PROOF: Let $T : L_1([01], \sum, \mu) \to X^*$ be a bounded linear operator and X^* has a weak Radon Nikodym property. Then there is a Pettis density ϕ from $[\mathbf{01}]$ into X^*, that is, T has a Pettis density valued in X^*. Clearly T has a Pettis density valued in X^{**}, therefore X^* has the weak* Radon Nikodym property. Conversely, let X^* have the weak* Radon Nikodym property. BY Proposition 1.8, X^* has the compact range property. Also, by Tneorem 2.3, we abtain that X^* has weak Radon Nikodym property.

The following Corollary 2.5 can be abtained from the Theorem 2.3 and the Theorem 2.4.

COROLLARY 2.5. *Let X be a Banach space. Then the following statements are equivalent.*

(a) *The dual space X^* has the weak Radon Nikodym property.*
(b) *The dual space X^* has the weak* Radon Nikodym property.*
(c) *The dual space X^* has the compact range property.*

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