JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 3, June 1990

A Note on Gronwall's Inequality

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ABSTRACT. We obtain one simple generalization of the Gronwall's inequality by using the Viswanatham's theorem.

1. Introduction

The celebrated Gronwall's inequality and its generalizations have been of vital importance in the study of existence, uniqueness, continuous dependence, comparison, perturbation, boundedness, and stability results of ordinary differential equations and integral equations.

Chu and Metcalf [**2**] obtained one generalization of the Gronwall's inequality by using the resolvent kernel. Akinyele [**1**] established some nonlinear integral inequalities in n-independent variables. Also, Yang [**5**] derived discrete generalizations of the Gronwall's inequality which have wide application in the study of finite difference equations and numerical analysis.

In this short note we obtain one simple generalization of the Gronwall's inequality by using the Viswanatham's theorem which is appeared in Theorem 1.9.2 of [**4**].

2. A Simple Generalization

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space with the Euclidean norm. For a fixed point $x_0 \in \mathbf{R}^n$, we denote $\overline{B_b(x_0)} = \{x \in \mathbf{R}^n : |x - x_0| \leq b\}$, where $b > 0$. We consider a nonlinear differential system

$$\text{(N)} \qquad x' = f(t,x), \qquad x(t_0) = x_0$$

defined by the continuous function $f : [t_0, t_0 + a] \times \overline{B_b(x_0)} \to \mathbf{R}^n$, where $t_0, a \in \mathbf{R}$.

Received by the editors on June 30, 1990.
1980 *Mathematics subject classifications*: Primary 34A40.

In order to prove the existence of a sequence of approximate solutions to a solution of (N) the notion of an equicontinuous set of functions is required. A set of functions $\mathcal{F} = \{f\}$ defined on a compact subset $K \subset \mathbf{R}^n$ is said to be *equicontinuous* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $|x_1 - x_2| < \delta$ with $x_1, x_2 \in K$ implies $|f(x_1) - f(x_2)| < \varepsilon$ for all $f \in \mathcal{F}$. The fundamental property of such sets of functions needed here is given in the following lemma:

LEMMA 1 (Ascoli). *Let $\mathcal{F} = (f_n)$ be a sequence of functions from a compact subset $K \subset \mathbf{R}^n$ into $\mathbf{R}^n$. Assume that*

(i) *$\mathcal{F}$ is equicontinuous,*
(ii) *$\mathcal{F}$ is uniformly bounded, i.e., for every $f_n \in \mathcal{F}$ there is a constant M such that $|f_n(x)| \leq M$ for all $x \in K$.*

Then there exists a function $f \in \mathcal{F}$ such that $\mathcal{F}$ is uniformly converges to f on K.

The following is the well-known Cauchy-Peano existence theorem. For the proof see Theorem 1.1.2 in [4].

THEOREM 2. *Suppose that for every $(t, x) \in [t_0, t_0 + a] \times \overline{B_b(x_0)}$, $|f(t, x)| \leq M$ for some constant M. Then there exists at least one solution $x(t)$ of (N) on an interval $J = [t_0, t_0 + \alpha]$, where $\alpha = \min\{a, b/M\}$.*

Consider the system (N) defined by a continuous function $f : U \to \mathbf{R}$, where U is an open subset of $\mathbf{R}^2$. A solution $f(t)$ is a *maximal solution* of (N) on $J = [t_0, t_0 + a)$ if for every solution $x(t)$ on J, $x(t) \leq r(t)$ for all $t_0 \leq t < t_0 + a$.

THEOREM 3. *Assume that $f : [t_0, t_0 + a] \times \overline{B_b(x_0)} \to \mathbf{R}$ is continuous and for every $(t, x) \in [t_0, t_0 + a] \times \overline{B_b(x_0)}$, $|f(t, x)| \leq M$ for some constant M. Then the system (N) has a maximal solution on $[t_0, t_0 + \alpha)$, where $\alpha = \min\{a, b/(2M + b)\}$.*

PROOF: See Theorem 1.3.1 in [4].

An important technique in the theory of differential equations is the following comparison theorem:

THEOREM 4 [4]. *Let f be a real valued continuous function defined on an open subset U of $\mathbf{R}^2$. Suppose that there exists a maximal* *solution $r(t)$ of (N) on $[t_0, t_0+a)$. If $\phi : [t_0, t_0+a) \to \mathbf{R}$ is a continuous function satisfying*

(i) $(t, \phi(t)) \in U$, $t_0 \le t < t_0 + a$,
(ii) $\phi(t_0) \le x_0$,
(iii) $D\phi(t) \le f(t, \phi(t))$, $t_0 \le t < t_0 + a$,

where $D\phi(t)$ is the Dini derivative [**4**], *then we have*

$$\phi(t) \le r(t), \qquad t_0 \le t < t_0 + a.$$

Now, the celebrated Gronwall's inequality is the following:

THEOREM 5. *Let ϕ and ψ be continuous functions from $[t_0, t_0 + a)$ into the nonnegative real line $\mathbf{R}^+$. If*

$$\phi(t) \le M + \int_{t_0}^{t} \phi(s)\psi(s)\, ds, \qquad t_0 \le t < t_0 + a,$$

where M is a constant, then

$$\phi(t) \le M \exp\left(\int_{t_0}^{t} \psi(s)\, ds \right).$$

PROOF: See Theorem 1.9.1 of [**4**].

The following theorem [**4**] is the consequence of the comparison theorem (Theorem 4).

THEOREM 6 (Viswanatham). *Let $f : U \to \mathbf{R}$ be a continuous function defined on an open subset $U \subset \mathbf{R}^2$ and suppose that for every $(t, x) \in U, f$ is monotonic nondecreasing in x for all t. If $\phi : [t_0, t_0 + a) \to \mathbf{R}$ is a continuous function with the property*

(i) $(t, \phi(t)) \in U$, $t_0 \le t < t_0 + a$,
(ii) $\phi(t_0) \le x_0$,
(iii) $\phi(t) \le \phi(t_0) + \int_{t_0}^{t} f(s, \phi(s))\, ds$, $t_0 \le t < t_0 + a$.

Then

$$\phi(t) \le r(t), \qquad t_0 \le t < t_0 + a,$$

where $r(t)$ is the maximal solution of (N).

In view of the above theorems we obtain a generalization of Theorem 6.

THEOREM 7. *Let $f : [t_0, t_0 + a] \times \overline{B_b(x_0)} \to \mathbf{R}$ be continuous and assume that for each $(t, x) \in [t_0, t_0 + a] \times \overline{B_b(x_0)}$, f is monotonic nondecreasing in x for all t. If $\phi : [t_0, t_0 + a] \to \mathbf{R}$ is a continuous function such that*

$$\phi(t) \leq x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds, \quad t_0 \leq t < t_0 + a,$$

then we have

$$\phi(t) \leq r(t), \qquad t_0 \leq t < t_0 + a,$$

where $r(t)$ is the maximal solution of the system (N).

PROOF: Note that $r(t)$ exists by Theorem 3. We define

$$\phi_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \phi_n(s))\, ds.$$

By Theorem 2 these successive approximation exist on $[t_0, t_0 + \alpha]$, where $\alpha = \min\{a, b/M\}$. Putting $\delta = \varepsilon/M > 0$, we have

$$\begin{aligned} |\phi_n(t_1) - \phi_n(t_2)| &= \left| \int_{t_1}^{t_2} f(s, \phi_{n-1}(s))\, ds \right| \\ &\leq \int_{t_1}^{t_2} |f(s, \phi_{n-1}(s)|\, ds \\ &\leq |t_2 - t_1| M < \varepsilon \end{aligned}$$

if $|t_2 - t_1| < \delta$. This implies that the sequence $(\phi_n(t))$ is equicontinuous. Moreover, $(\phi_n(t))$ is uniformly bounded since

$$|\phi_n(t)| \leq |x_0| + M|t_2 - t_1| \leq |x_0| + M\alpha.$$

Also, it is monotonic nondecreasing since

$$\phi_{n-1}(t) - \phi_n(t) = \int_{t_0}^{t} [f(s, \phi_n(s)) - f(s, \phi_{n-1}(s)]\, ds \geq 0$$

whenever $\phi_n(t) \geq \phi_{n-1}(t)$. Therefore we have

$$\phi_n(t) \to \psi(s) = x_0 + \int_{t_0}^{t} f(s, \psi(s))\, ds$$

by Ascoli's lemma. It is clear that $\psi(t)$ is a solution of the system (N).

Finally, $\phi(t) \geq r(t)$ since $\phi(t) \geq \psi(t)$ for all $t_0 \leq t \leq t \leq t + \alpha$.

COROLLARY 8. *Under the hypotheses of Theorem 7, if*

$$\phi(t) \leq \psi(t) + \int_{t_0}^{t} f(s, \phi(s))\, ds,$$

then

$$\phi(t) \leq \psi(t) + r(t), \qquad t \geq t_0,$$

where $r(t)$ is the maximal solution of the system

$$x' = f(t, t + \psi(t)), \qquad x(t_0) = 0,$$

whenever $r(t)$ exists.

PROOF: We define $\eta(t) = \phi(t) - \psi(t)$. Then

$$\eta(t) \leq \int_{t_0}^{t} f(s, \eta(s) + \psi(s))\, ds.$$

Thus we have $\eta(t) \leq r(t)$ by Theorem 7. It follows that

$$\phi(t) \leq \psi(t) + r(t).$$

COROLLARY 9. *The Gronwall's inequality (Theorem 5) is a special case of Theorem 7.*

PROOF: If we set

$$f(t, x) = |f(t)|x, \quad t_0 = 0 \quad \text{and} \quad x_0 = M,$$

then the Gronwall's inequality is obtained.

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