JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 3, June 1990

An Application of Bredikhin's Theorem

S. HAHN AND Y. OH

ABSTRACT. We apply Bredikhin's theorem to the distribution of prime numbers in arithmetic progressions.

Let P denote the set of all prime numbers. Let S be a subset of P. Let G denote the multiplicative semigroup generated by S. We define two functions π_G and ν_G by

$$\pi_G(x) = \sum_{p \le x,\, p \in S} 1 \quad \text{and} \quad \nu_G(x) = \sum_{n \le x,\, n \in G} 1.$$

We say S is regular if there are numbers $\tau \ge 0$, $\varepsilon > 0$ so that

$$\pi_G(x) = \tau \frac{x}{\log x} + O\left(\frac{x}{\log^{1+\varepsilon} x}\right)$$

Suppose that S is regular. Then by a theorem of Bredikhin [4], there is a constant C_G so that

$$\nu_G(x) = C_G x \log^{\tau-1} x + O\left(\frac{x \log^{\tau-1} x}{\log\log^{\varepsilon_1} x}\right)$$

where $\varepsilon_1 = \min(1, \varepsilon)$. We say that S has Bredikhin density τ and Bredikhin number C_G. Trivially, P has Bredikhin density 1 and Bredikhin number 1. Let f_S be the function on P defined by

$$f_S(p) = \begin{cases} 1 & \text{if} \quad p \in S \\ 0 & \text{otherwise.} \end{cases}$$

Received by the editors on 29 June 1990.
1980 *Mathematics subject classifications*: Primary 11.

f_S determines a completely multiplicative arithmetic function which we denote by the same symbol f_S. Then by a theorem of Wirsing [4],

$$\begin{aligned}\nu_G(x) &= \sum_{n\le x} f_S(n)\\ &= \left(\frac{e^{-\gamma\tau}}{\Gamma(\tau)}+o(1)\right)\frac{x}{\log x}\prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\cdots\right)\end{aligned}$$

where γ is Euler's constant and $\Gamma(\tau)$ denotes the gamma-function. So

$$\begin{aligned}&C_G x\log^{\tau-1}x+O\left(\frac{x\log^{\tau-1}x}{\log\log^{\epsilon_1}x}\right)\\ &=\left(\frac{e^{-\gamma\tau}}{\Gamma(\tau)}+o(1)\right)\frac{x}{\log x}\prod_{p\le x,\,p\in S}\left(\frac{1}{1-\frac{1}{p}}\right).\end{aligned}$$

Hence we have

THEOREM 1. *Suppose that S is regular with τ and C_G. Then*

$$\prod_{p\le x,\,p\in S}\left(1-\frac{1}{p}\right)=\left(\frac{e^{-\gamma\tau}}{C_G\Gamma(\tau)}+o(1)\right)\frac{1}{\log^{\tau}x}$$

PROOF: We know that

$$(C_G+o(1))\frac{x}{\log x}\log^{\tau}x=\left(\frac{e^{-\gamma\tau}}{\Gamma(\tau)}+o(1)\right)\frac{x}{\log x}\prod_{p\le x,\,p\in S}\frac{1}{\left(1-\frac{1}{p}\right)}$$

From this the claim easily follows.

REMARK: The case when $S=P$ was proved by Merten.

Let $H=P-S$. Then we have, by Merten's result,

$$\frac{e^{-\gamma\tau}}{C_G\Gamma(\tau)}\frac{e^{-\gamma(1-\tau)}}{C_H\Gamma(1-\tau)}=e^{-\gamma}.$$

So $C_GC_H\Gamma(\tau)\Gamma(1-\tau)=1$, $C_GC_H=(\sin\pi\tau)/\pi$. Here we see that there may exist a theory similar to that of special values of gamma function and their relations [3]. Let n be an odd integer and a an integer such that $\gcd(n, a) = 1$. Let S denote the set $\{p \in P : p \equiv a \bmod n\}$. We know that S is regular with $\tau = 1/\varphi(n)$ and some $\varepsilon > 0$ [2]. Let $c(n, a)$ denote the Bredikhin number of S. Then

$$\prod_{(n,a)=1} \frac{e^{-\gamma/\varphi(n)}}{c(n,a)\Gamma(1/\varphi(n))} = e^{-\gamma}.$$

So we have

THEOREM 2.

$$\prod_{(n,a)=1} c(n,a) = \Gamma(1/\varphi(n))^{-\varphi(n)}.$$

So we are naturally led to the following computational

QUESTION 3: Is $c(4,1) = c(4,3)$?

REFERENCES

[1] Lars Ahlfors, "Complex Analysis," second edition, McGraw-Hill Book Company, 1966.
[2] T. Estermann, "Introduction to Modern Prime Number Theory," Cambridge University Press, 1952.
[3] N. Koblitz and A. Ogus, *Algebraicity of some products of values of the* Γ *function*, in "Automorphic Forms, Representations, and L-functions," part 2, AMS, 1979, pp. 343–346.
[4] A. Postnikov, "Introduction to Analytic Number Theory," AMS, 1988.

Department of Mathematics
KAIST
Taejon 305-701, Korea