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Fixed Point Theorems for Multivalued Mappings in Banach Spaces

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ABSTRACT. Let K be a nonempty weakly compact convex subset of a Banach space X and $T : K \to C(X)$ a nonexpansive mapping satisfying $P_T(x) \cap \mathrm{cl}\, I_K(x) \neq \emptyset$. We first show that if $I - T$ is semiconvex type then T has a fixed point. Also we obtain the same result without the condition that $I - T$ is semiconvex type in a Banach space satisfying Opial's condition. Lastly we extend the result of [5] to the case, that T is an 1-set contraction mapping.

I. Introduction

In 1965, F.E. Browder [3] and W.A. Kirk [10] proved that every nonexpansive T from a weakly compact convex subset K of a uniformly convex Banach space X into K has a fixed point (one may see Goebel and Reich [9] for more references related our subject). Later, T.C. Lim [13] extended this result for the case that T is multivalued and K. Deimling [6] proved the same result for the case that T is a condensing inward mapping. Furthermore, in [5], T.H. Chang and C.L. Yen proved that if T is a mapping from a weakly compact convex subset K of a Banach space X into the family of nonempty compact subsets of X satisfying $Tx \subset \mathrm{cl}\, I_K(x)$ for each x in K where $I_K(x) = \{(1-\lambda)x + \lambda y\, ;\, y \in K,\ \lambda \geq 0\}$ and $\mathrm{cl}\, I_K(x)$ is its closure, and if $I - T$ is semiconvex type, that is, for all x, y in K, $0 \leq \lambda \leq 1$, $u = \lambda x + (1-\lambda)y$ we have

$$d(u, Tu) \leq \varphi(\max[d(x, Tx), d(y, Ty)],$$

where $\varphi : R^+ \to R^+$ is nondecreasing, continuous from the right at 0 with $\varphi(0) = 0$ (here R^+ is the set of nonnegative real numbers), then T has a fixed point.

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In this paper we obtain fixed point theorems of mappings satisfying certain inwardness conditions, which properly contains [**5**], [**15**]. Also we obtain the same result without the condition that $I - T$ is semi-convex type in a Banach space satisfying Opial's condition. Lastly we extend the result of [**5**] to the case that T is an 1-set contraction mapping.

Recall that for bounded sets A and B in a metric space M, we set

$$H_0(A,B) = \sup\{d(x,B)\,;\, x \in A\}$$

and define the Hausdorff metric by

$$H(A,B) = \max\{H_0(A,B), H_0(B,A)\}.$$

A multivalued mapping $T : M \to 2^M$ is said to be a contraction mapping if there exists a constant $k \in (0,1)$ such that $H(Tx,Ty) \le k\,d(x,y)$, x, $y \in M$ and a nonexpansive mapping if $H(Tx,Ty) \le d(x,y)$, x, $y \in M$, where 2^M is the family of all nonempty closed subset of M and $C(M)$ the family of nonempty compact subsets of M.

II. Some fixed point theorems

A subset K of a Banach space X is said to be proximal if for each x in X there exists an element k in K for which $d(x,k) = d(x,K)$. Obviously every compact subset of X is proximal. Therefore we can define $P_T(x)$ which is a subset of Tx such that for any $y \in P_T(x)$, $d(x,y) = d(x,Tx)$, where T is a mapping from K into $C(X)$.

Now we state our first result.

THEOREM 1. *Let K be a nonempty closed convex subset of a Banach space X. Suppose that T is a contraction mapping from K into $C(X)$ satisfying $P_T(x) \cap \operatorname{cl} I_K(x) \neq \emptyset$, for all $x \in K$. Then T has a fixed point in K.*

The key to our approach in proving Theorem 1 is an application of the following lemma.

LEMMA 1. *Let M be a complete metric space and $g : M \to M$ an arbitrary mapping. Suppose there exists a lower semi continuous mapping φ of M into the nonnegative real numbers such that for each $x \in M$,*

$$d(x, g(x)) \leq \varphi(x) - \varphi(g(x)).$$

Then g has a fixed point in M.

The formulation of Lemma 1 came as an outgrowth of Caristi's study [4] of fixed point theory for the contraction mappings.

We shall also need the following lemma. An explicit proof of the lemma is given in [4].

LEMMA 2. *Let K be a convex subset of a normed linear space. Then $x - y \in \operatorname{cl} I_K(x)$ if and only if*

$$\lim h^{-1} d(x - hy, K) = 0.$$

PROOF OF THEOREM 1: Assuming that T has no fixed points we may clearly suppose $d(x, Tx) > 0$ for each $x \in K$. By the condition there exists a constant $k \in (0,1)$ such that $H(Tx, Ty) \leq k\, d(x,y)$, x, $y \in K$. So we can select $\varepsilon > 0$ so that $k < (1-\varepsilon)(1+\varepsilon)^{-1}$. For given $x \in K$ we choose $z \in P_T(x) \cap \operatorname{cl} I_K(x)$. Then by Lemma 2 there exists $h \in (0,1)$ such that

$$h^{-1} d((1-h)x + hz, K) < \varepsilon d(x, Tx). \tag{1}$$

Writing $z_0 = (1-h)x + hz$, we observe that $\|z_0 - x\| = h\|z - x\|$. And by (1) there exists $y \in K$, $y \neq x$, such that

$$\|z_0 - y\| < h\varepsilon d(x, Tx) \tag{2}$$

and thus

$$\begin{aligned} \|x - y\| / \|z_0 - x\| &\leq [\|x - z_0\| + \|z_0 - y\|] / \|z_0 - x\| \\ &= 1 + \|z_0 - y\| / \|z_0 - x\| \\ &< 1 + \|z_0 - y\| / h d(x, Tx) \\ &< 1 + \varepsilon. \end{aligned}$$

Therefore

$$(1+\varepsilon)^{-1}\|x-y\| < \|z_0 - x\|. \tag{3}$$

Combining (2), (3), and using the definition of z_0 along with the fact that $z \in P_T(x)$ we obtain:

$$\begin{aligned}
d(y,Ty) &\leq \|y-z_0\| + d(z_0,Tx) + k\|x-y\| \\
&= \|y-z_0\| + d(x,Tx) - \|x-z_0\| + k\|x-y\| \\
&< \varepsilon\|x-z_0\| + d(x,Tx) - \|x-z_0\| + k\|x-y\| \\
&= d(x,Tx) + k\|x-y\| - (1-\varepsilon)\|x-z_0\| \\
&< d(x,Tx) + k\|x-y\| - (1-\varepsilon)(1+\varepsilon)^{-1}\|x-y\| \\
&= d(x,Tx) + [k-(1-\varepsilon)(1+\varepsilon)^{-1}]\|x-y\|.
\end{aligned}$$

Letting $\eta = -[k-(1-\varepsilon)(1+\varepsilon)^{-1}]$, the above reduces to :

$$\eta\|x-y\| \leq d(x,Tx) - d(y,Ty)$$

with $\eta > 0$. We now define $g : K \to K$ by taking $g(x) = y$ with y determinded as above, and let $\varphi(x) = \eta^{-1}d(x,Tx)$. So Lemma 1 implies the existence of $x_0 \in K$ such that $x_0 = g(x_0)$. But $g(x) = y \neq x$ for all $x \in K$ by definition, and our assumption that T has no fixed points is contradicted.

Note that Theorem 1 and the following theorem properly include Downing and Kirk [**8**] and Chang and Yen [**5**], respectively.

THEOREM 2. *Let K be a nonempty weakly compact convex subset of a Banach space X and $T : K \to C(X)$ a nonexpansive mapping satisfying $P_T(x) \cap \operatorname{cl} I_K(x) \neq \emptyset$ for each x, and let $I-T$ be semiconvex type. Then T has a fixed point.*

PROOF: It suffices to show (see Theorem 1 in [**5**]) that

$$\inf\{d(x,Tx)\,;\, x \in K\} = 0.$$

For given x_0 in $K$and $t \in [0,1)$ we define $T_t : K \to C(X)$ by $T_t x = (1-t)x_0 + tTx$ for all $x \in K$. Then by Theorem 1 T_t has a fixed point x_t. Hence there is a $y_t \in Tx_t$ such that

$$x_t = (1-t)x_0 + ty_t$$

so

$$\|x_t - y_t\| \leq ((1-t)/t)\|x_t - x_0\|$$

and

$$d(x_t, Tx_t) = \inf\{\|x_t - y\| \,;\, y \in Tx_t\}$$
$$\leq \|x_t - y_t\| \leq ((1-t)/t)\|x_t - x_0\|.$$

Since the set $\{\|x_t - x_0\| \,;\, 0 \leq t \leq 1\}$ is bounded, we get

$$0 \leq \inf_{x \in K} d(x, Tx) \leq \inf_{t \in [0,1)} d(x_t, Tx_t) \leq \inf_{t \in [0,1)} ((1-t)/t)\|x_t - x_0\| = 0.$$

Hence

$$\inf\{d(x, Tx) \,;\, x \in K\} = 0.$$

In [**14**] Opial observed that every uniformly convex Banach space which possesses a weakly continuous duality mapping satisfies the condition.

(A) If the sequence $\{x_n\}$ is weakly convergent to x_0 and if $x \neq x_0$ then

$$\liminf \|x_n - x\| > \liminf \|x_n - x_0\|.$$

We say that a Banach space satisfies Opial's condition if it has property (A). Such spaces include Hilbert spaces and the space ℓ^p, $1 < p < \infty$.

In the following theorem we obtain the same result in a Banach space satisfying Opial's condition in spite of deleating the condition that $I - T$ is semiconvex type.

THEOREM 3. *Let K be a weakly compact convex subset of a Banach space X with Opial's condition and $T : K \to C(X)$ a nonexpansive mapping satisfying $P_T(x) \cap \operatorname{cl} I_K(x) \neq \emptyset$ for each x in K. Then T has a fixed point.*

PROOF: Let $a \in K$. Let $\{\lambda_n\}$ be a decreasing sequence of positive numbers less then 1 and $\lim \lambda_n = 0$. For each n, the mapping $T_n : K \to C(X)$ defined by $T_n(x) = \lambda_n a + (1 - \lambda_n)Tx$ is a contraction mapping and hence has a fixed point x_n by Theorem 1. Thus $x_n \in \lambda_n a + (1 - \lambda_n)Tx_n$ and there exists a $y_n \in Tx_n$ with $x_n = \lambda_n a +$

$(1-\lambda_n)y_n$. Since K is bounded, we have $\|x_n - y_n\| \to 0$. Since K is weakly compact, we may assume that x_n converges weakly to some element x in K. For each n, let $p_n \in Tx$ be chosen such that

$$\|y_n - p_n\| \leq \|x_n - x\|.$$

Since Tx is a compact subset of X, there exists a convergent subsequence, say also $\{p_n\}$ in Tx with $\lim p_n = p \in Tx$. Therefore we get

$$\liminf \|x_n - p\| = \liminf \|y_n - p\| \leq \liminf \|x_n - x\|.$$

Hence $x = p \in Tx$.

Let (K, d) be a complete metric space. We define the Kuratowski measure of noncompactness as a nonnegative real valued function on set of all bounded subsets of K such that $\alpha(D) = \inf\{r > 0\,;\, D$ is covered by finitely many sets with diameter less than $r\}$. It is well-known that $\alpha(D) = 0$ if and only if the closure of D is compact.

A mapping $T : K \to K$ is said to be condensing if, for each bounded subset D of K, TD is bounded and $\alpha(TD) < \alpha(D)$ for all $\alpha(D) \neq 0$ where $T(D) = \bigcup_{x \in D} Tx$. A mapping $T : K \to 2^K$ is said to be a k-set contraction if T is continuous and bounded, and there is a number $k \geq 0$ such that $\alpha(T(D)) \leq k\alpha(D)$ for all bounded subset D of K.

LEMMA 3 (K. Deimling [6]). *Let K be a closed bounded convex subset of a Banach space X and $T : K \to 2^X$ a condensing and weakly inward mapping on K (that is, $Tx \subset \operatorname{cl} I_K(x)$). Then T has a fixed point.*

Now we have the following theorem which generalizes a result of [5].

THEOREM 4. *Let K be a nonempty weakly compact convex subset of a Banach space X and let $T : K \to 2^X$ be 1-set contraction satisfying weakly inwardness and let $I - T$ be semiconvex type. Then T has a fixed point.*

PROOF: According to Theorem 1 in [5], it suffices to show

$$\inf\{d(x, Tx)\,;\, x \in K\} = 0.$$

Let $x_0 \in K$ and for $0 \le t < 1$ define $T_t : K \to 2^X$ by $T_t(x) = (1-t)x_0 + tTx$ for all $x \in K$. Then T_t is a t-set contraction with $t < 1$, hence it has a fixed point x_t in K. That is, there exists a $y_t \in Tx_t$ such that

$$x_t = (1-t)x_0 + ty_t$$

so

$$\|x_t - y_t\| \le (1-t)\|x_0 - y_t\|$$

and

$$\begin{aligned} d(x_t, Tx_t) &= \inf\{\|x_t - y\|\,;\, y \in Tx_t\} \\ &\le \|x_t - y_t\| \\ &\le (1-t)\|x_0 - y_t\|. \end{aligned}$$

Since $\{x_0 - y_t\,;\, 0 \le t < 1\}$ is bounded, we have

$$0 \le \inf_{x \in K} d(x, Tx) \le \inf_{t \in [0,1)} d(x_t, Tx_t) \le \inf_{t \in [0,1)} (1-t)\|x_0 - y_t\| = 0.$$

Hence we complete the proof.

Note that every single valued nonexpansive mapping is 1-set contraction, but the multivalued nonexpansive mapping $T : X \to 2^X$ need not to be 1-set contraction unless Tx is compact. Therefore we have the following

COROLLARY (Theorem 9 in [**5**]). *Let K be a nonempty weakly compact convex subset of a Banach space X and $T : K \to C(X)$ nonexpansive satisfying $Tx \subset \operatorname{cl} I_K(x)$ for each x in K, and let $I - T$ be semiconvex type. Then T has a fixed point.*

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