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A Note on Basic Construction and Index for Subfactors

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ABSTRACT. The converse of Pimsner-Popa theorem is proved, and represent the index $[M : N]$ by finite sum of elements of M.

1. Introduction

Let M be a finite factor with normalized trace tr and $N \subset M$ a subfactor. In a recent paper [**1**], V.F.R. Jones considered the coupling constant of N in his representation on $L^2(M, tr)$ as an invariant for N up to conjugations by automorphisms of M. He call this invariant the index of N in M and denotes it $[M : N]$ and also V.F.R. Jones shows that if M is a factor of type II_1 the index is more than 4 or take values $4\cos^2(\pi/n)$ for $n \geq 3$. If this index is finite then the trace preserving conditional expectations of M onto N, regarded as an operator on $L^2(M)$, generates together with M a finite factor M_1 this factor is called in Jones' terminology the extension of M by N and the construction of M_1 from M and N the basic construction, the pair $M \subset M_1$ has the remarkable property that $[M_1 : M] = [M : N]$. In [**3**], if $N \subset M$ are finite factors with $[M : N] < \infty$, M. Pimsner and S. Popa construct the Pimsner-Popa basis and using this represent elements of M. In this paper we consider some properties of a pair of factors of type II_1 with $[M : N] < \infty$ and the converse of the theorem [**3**] 1.3 in which Pimsner-Popa basis was constructed and representation of elements of M using Pimsner-Popa basis.

2. Basic construction and index for subfactors

Throughout this paper M will be a finite factor with the normalized trace tr. We denote $\|x\|_2 = tr(x^*x)^{1/2}$, $x \in M$, the Hilbert norm given by tr and denote $L^2(M, tr)$ the Hilbert space completion of M

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in this norm. The canonical conjugation of $L^2(M, tr)$ is denoted by J. It acts on M and $L^2(M, tr)$ by $Jx = x^*$ and satisfies $JMJ = M'$.

Let $N \subset M$ will denote a subfactor of M with $1_M = 1_N$ and E_N be the unique normal trace preserving conditional expectation of M onto N. Then, in fact, E_N is just the restriction to $M \subset L^2(M, tr)$ of the orthogonal projection e_N of $L^2(M, tr)$ onto $L^2(N, tr)$, the closure of N in $L^2(M, tr)$. The conditional expectation E_N and the projection e_N and the conjugation J are related by the properties;

1) If $x \in M$ then $x \in N$ if and only if $e_N x = x e_N$
2) $e_N x e_N = E_N(x) e_N$, $x \in M$
3) J commutes with e_N

If M is a finite factor and acts on the Hilbert space H, the Murray and von Neumann coupling constant $\dim_N H$ is defined as $tr([M'\xi])/tr \cdot ([M\xi])$, where tr' be the canonical trace on M' and $0 \neq \xi \in H$ and for $A \subset B(H)$ $[A\xi]$ is the orthogonal projection onto the closed subspace generated by $A\xi$. It is shown in [**4**] that this definition is independent of $\xi \neq 0$. For a pair of finite factors $N \subset M$ V.F.R. Jones defined in [**1**] the index of N in M, $[M : N]$, to be the number $\dim_N(H)/\dim_M(H)$ or equivalently $\dim_N L^2(M, tr)$. In particular $[M : N]$ is a conjugacy invariant for N as a subfactor of M. In the case $N \subset M$ comes from the group construction in [**2**], for some I.C.C discrete groups $G_0 \subset G$ then $[M : N]$ coincide with the index of G_0 in G. When M is the crossed product of N by some outer action of a discrete group K, $M = N \times K$, then $[M : N]$ is just the cardinal of K. The index $[M : N]$ has the following properties ([**1**]);

1. $[M : N] = 1$ if and only if $M = N$
2. $[M : N] \geq 1$
3. If $N \subset P \subset M$ then $[M : P][P : N] = [M : N]$
4. $[M : N] = \infty$ iff N' is of type II_∞ iff M_1 is of type II_∞
5. If $[M : N] < \infty$ then M_1 is a finite factor and $[M_1 : M] = [M : N]$

In basic construction the extension M_1 of M by N is constructed as a factor acting on $L^2(M)$. In generally M_1 can not act on $L^2(N)$ but if $[M : N]$ is finite then M_1 can act on $L^2(N)$.

THEOREM 1. *Let $N \subset M$ be II_1 factors. If $[M : N]$ is finite then there exists a faithful representation of M on $L^2(N)$.*

PROOF: Since $[M : N]$ is finite M' and N' are finite factors. In particular M is a II_1 factor acting on $L^2(M, tr)$. So there exists a projection p in M' such that $tr_{M'}(p) = [M : N]^{-1}$. Then N and M are $*$-isomorphic to the reduced von Neumann algebras N_p and M_p respectively. Now

$$\begin{aligned}\dim N_p(p(L^2(M,tr)) &= tr_{N'}(p)\dim_N(L^2(M,tr))\\ &= tr_{M'}(p)[M:N]\\ &= 1.\end{aligned}$$

So N_P acts standardly on $p(L^2(M, tr))$. Now let $\pi : N \to B(L^2(M, tr))$ be the faithful representation of N, i.e. $\pi(x)$ act as left multiplication by x on $L^2(N, tr_N)$. Then $\pi(N)$ also acts standardly on $L^2(N, tr_N)$. So that N_P and $\pi(N)$ are spatially isomorphic. thus there exists a unitary operator u from $p(L^2(M, tr))$ to $L^2(N, tr_N)$ such that $\pi(x) = ux_pu^*$, $x \in N$. Define $\phi(x) = ux_pu^*$, $x \in M$, then ϕ is a faithful representation of M on $L^2(N)$.

Let M be a factor of type II_1 and N a subfactor of M. If $[M : N]$ is finite then there exists a Pimsner-Popa basis $\{m_1\}_{1\le i\le n+1}$ of elements in M, with n equal to the integer part of $[M : N]$, satisfying the properties;

a) $E(m_j^*m_k) = 0$ if $j \neq k$
b) $E(m_j^*m_j) = 1$ if $1 \le j \neq n$
c) $E(m_{n+1}^*m_{n+1})$ is a projection of trace $[M : N] - n$

Any such family satisfies ;

d) $\sum m_je_Nm_j^* = 1$
e) $\sum m_jm_j^* = [M : N]$
f) Every element m in M has a unique decomposition $m = \sum m_iy_i$ with $y_i \in N$, $y_{n+1} \in E_N(m_{n+1}^*m_{n+1})N$.

In M_1 $\{\sum a_ie_Nb_i : a_i, b_i \in M\}$ forms a dense subalgebra ([3] 1.2). Now we can find more explicite description of M_1 using Pimsner-Popa basis.

PROPOSITION 2. *Let $N \subset M$ be factors of type II_1 and M_1 be the extension of M by N. Then $M_1 = \{\sum m_ia_i, m_je_Nm_j \mid a_{i,j} \in N\}$, where n is the integer part of $[M : N]$ and $\{m_i : 1 \le i \le n+1\}$ is the Pimsner-Popa basis of M over N.*

PROOF: Let $x \in M_1$. By the property d) of Pimsner-Popa basis we have

$$x = \sum m_i e_N m_i^* x \sum m_j e_N m_j^* = \sum m_i e_N m_i^* x m_j e_N m_j^*.$$

By ([**3**] 1.2) there exists elements $a_{i,j}$ of M such that $m_i^* x m_j e_N = a_{i,j} e_N$. So

$$x = \sum m_i e_N a_{i,j} e_N m_j.$$

Since $e_N x e_N = E_N(x) e_N$ for all x in M by the properties of basic construction we have

$$x = \sum m_i E_N(a_{i,j}) e_N m_j^*.$$

Using Pimsner-Popa basis we can see that M is isomorphic as a right N-module with $N^n \oplus E_N(m_{n+1})^* m_{n+1})N$ ([**3**] 1.4). Now let $N \subset M_i$, $i = 1, 2$, be factors of type II_1 with $[M_1 : N] = [M_2 : N]$. We consider the question whether M_1 and M_2 are isomorphic each other as right N-modules.

PROPOSITION 3. *Let $N \subset M_i$, $i = 1, 2$, be factors of type II_1. If $[M_1 : N] = [M_2 : N]$ and this index is an integer then M_1 and M_2 are isomorphic as right N-modules.*

PROOF: Let $n = [M : N]$, $\{m_i : i = 1, \ldots, n\}$ and $\{m_i' : i = 1, \ldots, n\}$ be Pimsner-Popa basis for M_1 and M_2 over N respectively. Then by [**3**] any elements m of M_1, m' of M_2 have unique representations as follows;

$$m = \sum m_i y_i, \qquad y_i \in N$$
$$m' = \sum m_i' y_i, \qquad y_i' \in N$$

Define $\phi(\sum m_i y_i) = \sum m_i' y_i$, then $\phi : M_1 \to M_2$ is a surjective N module homomorphism. If $\sum m_i' y_i = 0$ then by uniqueness $y_i = 0$. Thus ϕ is a right N module isomorphism.

In ([**3**] 1.3) M. Pimsner and S. Popa construct the Pimsner-Popa basis and give some properties of the pair $N \subset M$ of factors with $[M : N] < \infty$. The converse of ([**3**] 1.3) will represent $[M : N]$ using finite number of elements of M.

THEOREM 4. *Let $N \subset M$ be factors of type II_1 with $[M:N] < \infty$. Suppose that there exists a finite set $\{m_i\}_{1 \leq j \leq n+1}$ of elements of M which satisfy the followings;*

a) $E_N(m_j^* m_k) = 0$ *if* $j \neq k$
b) $E_N(m_j^* m_j) = 1$ *if* $1 \leq j \leq n$
c) $E_N(m_{n+1}^* m_{n+1})$ *is a projection of trace* t
d) $\sum m_j e_N m_j^* = 1$

then $[M:N] = \sum m_i m_i^* = n + t$ *and* $\{m_i\}$ *is the Pimsner-Popa basis.*

PROOF: Let M_1 be the extension of M by N and E_M be the trace preserving conditional expectation of M_1 onto M. Then $E_M(e_N) = [M:N]^{-1} I_M$ by ([**1**] 3.4.1). So we have

$$1 = E_M\left(\sum m_i e_N m_i^*\right) = \sum m_i m_i^* E_M(e_N)$$

thus $[M:N] = \sum m_i m_i^*$.

Let $p = E_N(m_{n+1}^* m_{n+1})$. Since tr is a $[M:N]^{-1}$ trace on M we have

$$tr(pe_N) = [M:N]^{-1} tr(p) = [M:N]^{-1} t.$$

So

$$tr\left(\sum E_N(m_i^* m_i) e_N\right) = (n+t)[M:N]^{-1}.$$

Now

$$tr(E_N(m_i^* m_i) e_N) = tr(e_N m_i^* m_i e_N) = tr(m_i e_N m_i^*),$$

so that

$$1 = tr\left(\sum m_i e_N m_i^*\right) = tr\left(\sum E_N(m_i^* m_i) e_N\right) = (n+t)[M:N]^{-1}.$$

By the proof of the previous theorem we can see that following corollary which represents index and elements of M.

COROLLARY. *Let $N \subset M$ be II_1 factors. If there exists a finite subset $\{m_i\}$ of M such that $\sum m_i e_N m_i^* = 1$ then $[M:N] = \sum m_i m_i^*$*

and every elements of M can be written by the form $\sum m_i y_i$ *for some* $y_i \in N$.

PROOF: By proof of theorem we have $[M : N] = \sum m_i^* m_i$. Let $m \in M$, then by the equation $e_N x e_N = E_N(x) e_N$ for all $x \in M$ we have

$$m e_N = \sum m_i e_N m_i^* m e_N = \sum m_i E_N(m_i^* m_i) e_N.$$

So that by ([**3**] 1.2) we have

$$m = \sum m_i E_N(m_i^* m_i) e_N.$$

REMARK: Let $N \subset M$ are II_1 factors with $[M : N] < \infty$. Choose a projection e in M with $tr(e) = [M : N]^{-1}$ and let $P = \{e\}' \cap N$ then P is a II_1 factor and M is an extension of N by P. V.F.R Jones said this process the downward basic construction ([**3**] 1.8). Let p be a projection in M and let $L = \{p\}' \cap N$ then in generally L can not be a factor. If L is a factor then we can see easily $[N : L] = tr(p)^{-1}$. Here we can consider the following problems concerning basic construction and index of subfactors.

1. Let $N \subset M$ are factors of type II_1. For which projection e in M, is $\{e\}' \cap N$ a factor and $[N : P] = tr(e)^{-1}$, where $P = \{e\}' \cap N$?
2. Let $N \subset M_i$ are factors of type II_1 with $[M_1 : N] = [M_2 : N]$. Does there exist a $*$-isomorphism $\phi : M_1 \to M_2$ such that $\phi|_N = \mathrm{id}_N$?
3. Let $N \subset M_i$ are factors of type II_1 with $[M_1 : N] < [M_2 : N]$. Does there exist a factor P with $N \subset P \subset M_2$ such that $[P : N] = [M_1 : N]$?
4. Let $N_i \subset M$ are factors of type II_1 with $[M : N_1] = [M : N_2]$. Does there exist an automorphism ϕ on M such that ϕ maps N_1 onto N_2?
5. Let $N_i \subset M$ are factors of type II_1 with $[M : N_1] < [M : N_2]$. Does there exist a factor P with $N_2 \subset P \subset M$ such that $[M : P] = [M : N_1]$?

In fact the problems 2, 3 and 4, 5 are dual each other.

REFERENCES

[1] V.F.R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–25.

[2] D. McDuffd, *Central sequences and the hyperfinite factor*, Proc. London Math. Soc. **21** (1970), 443–461.

[3] M. Pimsner and S. Popa, *Entropy and index for subfactors*, Ann. Sicent. Ec. Norm. Sup. **106** (1986), 57–106.

[4] S. Stratila and L. Zsido, "Lectures on von Veumann algebras," Abacus Press, 1979.

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