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A Note on Total Stability

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ABSTRACT. It is well-known that under suitable conditions, uniform asymptotic stability implies total stability. We prove this theorem of Malkin by using Liapunov-like functions and so our proof is a detailed version of Yoshizawa's proof.

Duboshin(1940) introduced the concept of stability under constantly acting perturbations which is called total stability. Total stability means that if the perturbation is not too large and if the system is not too far from the origin initially it will remain near the origin.

The problem of total stability treated in many books by Fink [3], Lakshmikantham and Leela [5], Yoshizawa [8] et al. Under a mild condition Kato [4] proved that for autonomous or periodic systems, if a bounded solution is uniformly asymptotic stable, then it is totally stable. Athanassov [2] gave sufficient conditions for total stability of sets of a general kind for autonomous systems.

Consider the nonautonomous system

$$\text{(N)} \qquad x' = f(t,x), \quad x(t_0) = x_0, \qquad t_0 \geq 0,$$

where $f : J \times B_\rho \to R^n$ is continuous, J is an interval containing t_0 and $B_\rho = \{x \in R^n : \|x\| < \rho, \rho > 0\}$. Here $\|\cdot\|$ denotes the Euclidean norm on R^n. Let $x(t) = x(t, t_0, x_0)$ be a solution of (N) through (t_0, x_0). Assume that $f(t,0) = 0$ for all $t \in J$, so that $x = 0$ is a trivial solution of (N) through $(t_0, 0)$.

The trivial solution $x = 0$ of (N) is said to be totally stable if for every $\varepsilon > 0$ and $t_0 \in J$, there exist two numbers $\delta_1 = \delta_1(\varepsilon) > 0$ and $\delta_2 = \delta_2(\varepsilon) > 0$ such that for every solution $x_{f+g}(t) = x_{f+g}(t, t_0, x_0)$ of the perturbed system

$$\text{(P)} \qquad x' = f(t,x) + g(t,x),$$

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where $g : J \times B_\rho \to R^n$ is continuous, the inequality

$$\|x_{f+g}(t, t_0, x_0)\| < \varepsilon$$

holds for all $t \geq t_0$ provided that $\|x_0\| < \delta_1$ and $\|g(t,x)\| < \delta_2$ for all $\|x\| < \varepsilon$ and $t \in J$.

One of well-known results about total stability is the following Malkin's theorem:

If $f(t,x)$ in (N) is Lipschitzian in x uniformly with respect to t on J and if the trivial solution $x = 0$ of (N) is uniformly asymptotically stable, then it is totally stable.

Massera [**6**] proved this result by use of characteristic exponents and Fink [**3**] proved by using Gronwall's inequality. Yoshizawa [**8**] used Liapunov-like functions to prove that result.

In this paper we prove the above result using Liapunov-like functions in several steps and so our approach is a detailed computation of Yoshizawa's method.

The definitions of the various stability concepts and the preliminary results about the Liapunov functions are presented in [**5**] and [**7**].

THEOREM. *For the system* (N), *suppose that* f *satisfies*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x - y\|$$

for (t,x), $(t,y) \in J \times B_\rho$, *where* $L(t) \geq 0$ *is continuous on* J *with*

$$\left| \int_t^{t+u} L(s)\, ds \right| \leq K|u|$$

for some constant K. *Then the trivial solution* $x = 0$ *of* (N) *is totally stable if it is uniformly asymptotically stable.*

PROOF: By [**5**, Theorem 3.6.9], there exists a continuous function $V : J \times B_\rho \to R^+$ with the properties

(i) $a(\|x\|) \leq V(t,x) \leq b(\|x\|)$ for some strictly monotone increasing continuous functions a, $b : [0, \rho) \to R^+$,

(ii) $|V(t,x) - V(t,y)| \leq M\|x - y\|$ for (t,x), $(t,y) \in J \times B_{\delta(\delta_0)}$ where $\delta(\varepsilon)$, δ_0 appear in the definition of uniform asymptotic stability [**5**] and M is a nonnegative constant,

(iii)

$$D^+V(t,x) = \limsup_{h\to 0+} \frac{1}{h}[V(t+h, x+hf(t,x)) - V(t,x)] \\ \le -c[V(t,x)]$$

for some strictly monotone increasing continuous function c on $[0,\rho)$.

We show that $\|x_{f,g}\| < \varepsilon$ for all $t \ge t_0$ provided that $\|x_0\| < \delta_1$ and $\|g(t,x)\| < \delta_2$ for some $\delta_1 = \delta_1(\varepsilon)$ and $\delta_2 = \delta_2(\varepsilon)$.

Let $\|x_0\| < \rho/2 < \varepsilon$.

Step 1. $\|x_{f+g}(t)\| < \varepsilon$ for $t_0 \le t \le t_0 + h$ with h sufficiently small. Note that since

$$x_{f+g}(t) = x_0 + \int_{t_0}^{t} f(s, x_{f+g}(s))\,ds + \int_{t_0}^{t} g(s, x_{f+g}(x))\,ds,$$

we have, for every $t \in [t_0, t_0+h]$ with h sufficiently small,

$$\|x_{f+g}(t)\| \le \|x_0\| + \int_{t_0}^{t} L(s)\|x_{f+g}(s)\|\,dx + h\eta, \tag{1}$$

where $\eta = \sup\{\|g(t,x)\| : t \ge 0, \|x\| \le \rho\}$. From the Gronwall's inequality, we obtain

$$\|x_{f+g}(t)\| \le (\|x_0\| + h\eta)e^{kh}.$$

This implies that if $t \in [t_0, t_0+h]$ and $\|x_0\| < \rho/2 < \varepsilon$, then $\|x_{f+g}(t)\| \le \rho/2 < \varepsilon$ for sufficiently small h.

Step 2. $\|x_{f+g}(t) - x_f(t)\| \le h\eta e^{kh}$ for $t_0 \le t \le t_0 + h$ with h sufficiently small : Since

$$x_f(t) = x_f(t, t_0, x_0) = x_0 + \int_{t_0}^{t} f(s, x_f(s))\,ds,$$

we have

$$\|x_{f+g}(t) - x_f(t)\| \\ = \left\| \int_{t_0}^{t} [f(s, x_{f+g}(s)) - f(s, x_f(s))]\,ds + \int_{t_0}^{t} g(s, x_{f+g}(s))\,ds \right\|.$$

Thus

$$\|x_{f+g}(t) - x_f(t)\| \leq \int_{t_0}^{t} L(s)\|x_{f+g}(s) - x_f(s)\|\, ds + h\eta$$

for $t_0 \leq t \leq t_0 + h$ with h sufficiently small. Hence, by the Gronwall's inequality we have

$$\|x_{f+g}(t) - x_f(t)\| \leq h\eta e^{kh}. \tag{2}$$

Step 3. $D^+V(t, x_{f+g}(t)) \leq -c(\|x_{f+g}(t)\|) + c[b^{-1}(\iota)]$: we have

$$\begin{aligned} &V(t+h, x_{f+g}(t+h)) - V(t+h, x_f(t+h)) \\ \leq &M|x_{f+g}(t+h) - x_f(t+h)| \end{aligned} \tag{3}$$

if $|x_f(t+h)| < \delta(\delta_0)$ and $|x_{f+g}(t+h)| < \delta(\delta_0)$. (3) becomes

$$V(t+h, x_{f+g}(t+h)) - V(t+h, x_f(t+h)) \leq Mh\eta e^{kh}$$

for sufficiently small h, by Step 2. Thus we have

$$\limsup_{h\to 0+} \frac{1}{h}[V(t+h, x_{f+g}(t+h)) - V(t+h, x_f(t+h))] \leq M\eta. \tag{4}$$

Hence

$$\begin{aligned} &\limsup_{h\to 0+} \frac{1}{h}[V(t+h, x_{f+g}(t+h)) - V(t, x_{f+g}(t))] \\ \leq &\limsup_{h\to 0+} \frac{1}{h}[V(t+h, x_f(t+h)) - V(t, x_{f+g}(t))] \\ &+ \limsup_{h\to 0+} \frac{1}{h}[V(t+h, x_{f+g}(t+h)) - V(t+h, x_f(t+h))] \\ \leq &- c(\|x_{f+g}(t)\|) + M\eta \end{aligned} \tag{5}$$

if $|x_{f+g}(t)| < \varepsilon$ for $t \in [t_0, t_0 + h]$, by the assumption (iii) and the formula (4). Thus in (5)

$$\begin{aligned} D^+V(t, x_{f+g}(t)) &\leq -c(\|x_{f+g}(t)\|) + M\eta \\ &\leq -c(\|x_{f+g}(t)\|) + M\delta_2 \\ &\leq -c(\|x_{f+g}(t)\|) + c[b^{-1}(\iota)]. \end{aligned}$$

since $\|g(t,x)\| < \delta_2$ and by putting $\delta_2(\varepsilon) = \frac{1}{M}c[b^{-1}(\iota)]$.

Step 4. $\|x_{f+g}(t)\| < \varepsilon$ for all $t \geq t_0$: We prove by contradiction. Suppose that there exists a $t_1 > t_0$ such that $\|x_{f+g}(t_1)\| \geq \varepsilon$. Then the exists a t_2 with $t_0 < t_2 \leq t_1$ such that

$$\|x_{f+g}(t_2)\| = \varepsilon \text{ and } \|x_{f+g}(t)\| < \varepsilon \text{ for all } t \in [t_0, t_2).$$

Put $\overline{V}(t) = V(t, x_{f+g}(t))$. Then by (i)

$$\begin{aligned}\overline{V}(t_2) = V(t_2, x_{f+g}(t_2)) &\geq a(\|x_{f+g}(t_2)\|)\\ &= a(\varepsilon) > \iota\end{aligned}$$

if $\iota < a(\varepsilon)$. Also, by (i)

$$\begin{aligned}\overline{V}(t_0) = V(t_0, x_0) &\leq b(\|x_0\|)\\ &< b(\delta_1) = b(b^{-1}(\iota)) = \iota\end{aligned}$$

if we take $\delta_1(\varepsilon) = b^{-1}\iota$. Thus there exists a t_3 with $t_0 < t_3 < t_2$ such that

$$\overline{V}(t_3) = \iota \text{ and } \overline{V}(t) > \iota \text{ for all } t \in (t_3, t_2). \tag{6}$$

By the condition (i), we have

$$a(\|x_{f+g}(t_3)\|) \leq V(t_3, x_{f+g}(t_3)) = \overline{V}(t_3) = \iota \leq b(\|x_{f+g}(t_3)\|).$$

This implies that

$$b^{-1}(\iota) \leq \|x_{f+g}(t_3)\| \leq a^{-1}(\iota) < \varepsilon$$

since $\iota < a(\varepsilon)$.

Now, by Step 3, we obtain

$$\begin{aligned}D^+\overline{V}(t_3) &= \limsup_{h \to 0+} \frac{1}{h}[\overline{V}(t_3 + h) - \overline{V}(t_3)]\\ &= \limsup_{h \to 0+} \frac{1}{h}[V(t_3 + h, x_{f+g}(t_3 + h)) - V(t_3, x_{f+g}(t_3))]\\ &\leq -c(\|x_{f+g}(t_3)\|) + c[b^{-1}(\iota)]\\ &\leq -c[b^{-1}(\iota)] + c[b^{-1}(\iota)] = 0.\end{aligned}$$

It follows that $\overline{V}(t) \leq \overline{V}(t_3)$ for all $t > t_3$, which contradicts to (6). This completes the proof.

REMARK: Athanassov's example [**1**, Remark 3] shows that the converse of Malkin's theorem does not hold in general. However, he proved a partial converse theorem [**1**, theorem 3.3] to the Malkin's theorem.

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