JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 3, June 1990

Exponential Asymptotic Stability in Perturbed Systems

SUNG KYU CHOI AND CHEONG SONG CHOI

ABSTRACT. In this paper we investigate the problem of exponential asymptotic stability (EAS) in perturbed nonlinear systems of the differential system $x' = f(t,x)$. Also, a simple method for constructing Liapunov functions is used to prove a kind of Massera type converse theorem.

1. Introduction

In this paper we study exponential asymptotic stability (EAS) of the following systems of ordinary differential equation:

$$x' = f(t,x) \tag{N}$$

and its perturbed systems

$$y' = f(t,y) + g(t,y), \tag{P_1}$$
$$y' = f(t,y) + g(t,y,Ty), \tag{P_2}$$
$$y' = f(t,y) + g(t,y) + h(t,y), \tag{P_3}$$
$$y' = f(t,y) + g(t,y) + h(t,y,Ty). \tag{P_4}$$

The notion of EAS in dynamical systems was introduced by Elaydi and Farran [4]. They investigated the properties of EAS dynamical systems on a compact Riemannian manifold, and gave some analytic criteria for an autonomous differential system and its perturbed system to be EAS.

Brauer[2] examined EAS for the trivial solution $y = 0$ of (P_1) by means of the variation of constants formula for nonlinear systems due to Alekseev.

Pachpatte[6] obtained an asymptotic behavior of (P_2) when the trivial solution of (N) is EAS.

Received by the editors on 20 June 1990.
1980 *Mathematics subject classifications*: Primary 34D10.

Athanassov[1] defined the global exponential stability in variation (GESV) and then showed that the existence of Liapunov function when the trivial solution of (N) is GESV. His method of proof is an adaptation of Theorem 3.6.1 in [5].

The purpose of this paper is to investigate the EAS for the trivial solution of (P_3) and (P_4) and prove the existence of Liapunov function when the trivial solution of (N) is generalized exponentially asymptotically stable in variation (GEASV) [5].

2. Preliminaries

Let the symbol $|\cdot|$ denote any convenient norm on the Euclidean n-space R^n and the corresponding norm for $n \times n$ matrices. We shall mean by $C(R^n, R)$ the space of continuous functions from R^n into R, with the sup norm. Let us consider the nonlinear differential system

$$\text{(N)} \qquad x' = f(t, x), \quad x(t_0) = x_0,$$

where $f \in C(J \times R^n, R^n)$ with $f(t, 0) = 0$ and $J = [0, \infty)$ containing t_0. If we assume that f has continuous partial derivatives $\partial f/\partial x$ on $J \times R^n$ and the solution $x(t) = x(t, t_0, x_0)$ of (N) through $(t_0, x_0) \in J \times R^n$ exists for $t \geq t_0 \geq 0$, then

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0)$$

exists and is the solution of the variational system

$$\text{(V)} \qquad z' = \frac{\partial}{\partial x} f(t, x(t, t_0, x_0))z$$

such that $\Phi(t_0, t_0, x_0)$ is the unit matrix [5].

First, we need the following Alekseev's nonlinear variation of constants formula for the perturbed nonlinear system

$$(\text{P}_1) \qquad y' = f(t, y) + g(t, y)$$

where $g \in C(J \times R^n, R^n)$.

LEMMA 2.1 [5, Theorem 2.6.3]. *If $\partial f/\partial x$ exists and is continuous on $J \times R^n$, then any solution $y(t,t_0,x_0)$ of* (P_1) *with $y(t_0) = x_0$ satisfies the integral equation*

$$y(t,t_0,x_0) = x(t,t_0,x_0) + \int_{t_0}^{t} \Phi(t,s,y(s,t_0,x_0))g(s,y(s,t_0,x_0))\,ds$$

for $t \geq t_0 \geq 0$ provided the solution $x(t,t_0,x_0)$ of (N) *exists for $t \geq t_0 \geq 0$.*

LEMMA 2.2 [5, Theorem 2.6.4]. *Let $\partial f/\partial x$ exist and be continuous on $J \times R^n$. Assume that $x(t,t_0,x_0)$ and $x(t,t_0,y_0)$ are the solutions of* (N) *through (t_0,x_0) and (t_0,y_0), respectively, existing for $t \geq t_0 \geq 0$, such that x_0, y_0 belong to a convex subset of R^n. Then for $t \geq t_0 \geq 0$,*

$$x(t,t_0,y_0) - x(t,t_0,x_0) = \left[\int_0^1 \Phi(t,t_0,x_0 + s(y_0 - x_0))\,dx\right] \cdot (y_0 - x_0).$$

LEMMA 2.3 [1, Lemma 2.1]. *Let D be a convex subset of R^n, let x_0, $y_0 \in D$, and let t be such that $x(t,t_0,x_0)$, $x(t,t_0,y_0) \in D$. Then*

$$|x(t,t_0,x_0) - x(t,t_0,y_0)| \leq |x_0 - y_0| \sup_{\eta \in D} |\Phi(t,t_0,\eta)|.$$

We recall some notions of stability.

DEFINITION 2.4: The trivial solution $x = 0$ of (N) is said to be

(i) *exponentially asymptotically stable* (EAS) if there exist constants $K > 0$, $c > 0$ such that

$$|x(t,t_0,x_0)| \leq K|x_0|e^{-c(t-t_0)}$$

for $t \geq t_0 \geq 0$ and $|x_0| < \infty$.

(ii) *exponentially asymptotically stable in variation* (EASV) if there exist constants $K > 0$, $c > 0$ such that

$$|\Phi(t,t_0,x_0| \leq Ke^{-c(t-t_0)}$$

for $t \geq t_0 \geq 0$ and $|x_0| < \infty$,

(iii) *generalized exponentially asymptotically stable in variation* (GEASV) if

$$|\Phi(t, t_0, x_0)| \leq K(t_0)e^{p(t_0)-p(t)}$$

for $t \geq t_0 \geq 0$ and $|x_0| < \infty$, where $K(t) > 0$ is continuous on J, $p \in \kappa$ for $t \in J$, and $p(t) \to \infty$ as $t \to \infty$. Here $p \in \kappa$ means $p \in C(J, R^+)$, $p(0) = 0$, and $p(t)$ is strictly increasing in $t \in J$.

Note that (iii) becomes (ii) when $K(t) \equiv K > 0$ and $p(t) = \alpha t$, $\alpha > 0$.

3. Main results

Brauer [**2**, Theorem 2] examined EAS for the trivial solution of (P_1) and obtained EAS for the trivial solution of

$$(P_2) \qquad y' = f(t, y) + g(t, y) + h(t, y),$$

where $h \in C(J \times R^n, R^n)$ with $h(t, 0) = 0$, as a corollary of his Theorem 2. We obtain an asymptotic behavior of solutions of (P_2).

THEOREM 1. *For the system* (P_2), *we assume the following conditions:*

(i) $|g(t, y)| = o(|y|)$ *as* $|y| \to 0$ *uniformly in* t,
(ii) *there exists an* $\alpha > 0$ *such that* $|x| < \alpha$ *and* $t \in J$ *imply* $|h(t, x)| \leq \gamma(t)$, *where* $\gamma \in C(J, R^+)$ *with* $\int_0^\infty \gamma(t)\, dt < \infty$.

If the trivial solution $x = 0$ *of* (N) *is EAS, then there exist* $T_0 \geq 0$ *and* $\delta > 0$ *such that* $t_0 \geq T_0$ *and* $|x_0| < \delta$ *imply every solution* $y(t)$ *of* (P_2) *tends to zero as* $t \to \infty$.

PROOF: By the assumption $|\Phi(t, t_0, x_0)| \leq Ke^{-c(t-t_0)}$ for some $K > 0$ and $c > 0$. We choose $T \geq 1$ and $\delta \leq \varepsilon$. Let $T_0 \geq T$ be so large that $t \geq T_0$ implies

$$\int_1^t \exp[-(c - K\varepsilon)(t - s)]\gamma(s)\, ds < \frac{\delta}{2k} = \delta_1.$$

This is possible by the fact

$$\lim_{t \to \infty} e^{-ct} \int_1^t e^{cs}\gamma(s)\, ds = 0$$

[**5**, Theorem 2.14.6].

Let $t_0 \geq T_0$ and $|y_0| < \frac{\delta}{2K} = \delta_1 < \delta$. Then, by Lemma 2.1, we have

$$\begin{aligned}|y(t)| &\leq |\Phi(t, t_0, y_0)|\,|y_0| + \int_{t_0}^{t} |\Phi(t, s, y(s)|\,|g(s, y(s)) + h(s, y(s))|\, ds \\ &\leq K|y_0|e^{-c(t-t_0)} + \int_{t_0}^{t} Ke^{-c(t-s)}[\varepsilon|y(s)| + \gamma(s)]\, ds.\end{aligned}$$

Thus

$$\begin{aligned}|y(t)|e^{ct} \leq\ & K|y_0|\exp(ct_0)\exp[K\varepsilon(t-t_0)] \\ &+ \int_{t_0}^{t} Ke^{cs}\gamma(s)\exp[K\varepsilon(t-s)]\, ds\end{aligned}$$

by the Gronwall's inequality. In other words, we have

$$|y(t)| \leq K\delta_1 \exp[-(c-K\varepsilon)(t-t_0)] + K\int_{t_0}^{t} \exp[-(c-K\varepsilon)(t-s)]\gamma(s)\, ds.$$

This inequality yields

$$\begin{aligned}|y(t)| &\leq K\delta_1 + K\int_{1}^{t} \exp[-(c - K\varepsilon(t-s)]\gamma(s)\, ds \\ &\leq K\delta_1 + \frac{\delta}{2} < \delta.\end{aligned}$$

i.e., $|y(t)| < \varepsilon$ holds on $[t_0, \infty)$. This implies that the above inequality is true for $t \geq t_0$. Hence $y(t) \to 0$ as $t \to \infty$.

Now, we consider the perturbed system

$$(\mathrm{P}_3) \qquad y' = f(t, y) + g(t, y) + h(t, y, Tt),$$

where $h \in C(J \times R^n, R^n)$ and $T : R^n \to R^n$ is a continuous operator. Pachpatte [**6**, Theorem 2] obtained an asymptotic behavior of the system

$$y' = f(t, y) + g(t, y, Ty).$$

As an adaptation of this, an asymptotic behavior of (P_3) is obtained. To do this we need an integral inequality which is a generalization of Pachpatte's inequality.

LEMMA 2. *If $u(t)$, $a(t)$, $b(t)$ and $c(t)$ are nonnegative continuous functions on $[0,\infty)$ with the property that*

$$u(t) \le u_0 + \int_{t_0}^{t} u(s)\,ds + \int_{t_0}^{t} a(s)u(s)\,ds + \int_{t_0}^{t} b(s)\int_{t_0}^{t} c(\tau)u(\tau)\,d\tau\,ds,$$

where u_0 is a nonnegative constant, then

$$u(t) \le u_0 \exp\left\{\int_{t_0}^{t}\left[1 + a(s) + b(s)\int_{t_0}^{t} c(\tau)\,d\tau\right]ds\right\}.$$

PROOF: Let

$$U(t) = u_0 + \int_{t_0}^{t} u(s)\,ds + \int_{t_0}^{t} a(s)u(s)\,ds + \int_{t_0}^{t} b(s)\int_{t_0}^{s} c(\tau)u(\tau)\,d\tau\,ds.$$

Then $u(t) \le U(t)$ and

$$\begin{aligned} U'(t) &= u(t) + a(t)u(t) + b(t)\int_{t_0}^{t} c(s)u(s)\,ds \\ &\le U(t) + a(t)U(t) + b(t)\int_{t_0}^{t} c(s)U(s)\,ds \\ &\le U(t)\left[1 + a(t) + b(t)\int_{t_0}^{t} c(s)\,ds\right]. \end{aligned}$$

Consequently we have

$$u(t) \le U(t) \le u_0 \exp\left\{\int_{t_0}^{t}\left[1 + a(s) + b(s)\int_{t_0}^{s} c(\tau)\,d\tau\right]ds\right\}.$$

THEOREM 3. *For the system* (P_3), *assume that*

(i) *$g(t,y) = o(|y|)$ as $|y| \to 0$ uniformly in t,*

(ii) *$|h(t,y,Ty)| \le h(t)\left[|y| + e^{-c_1 t}\int_{t_0}^{t}\mu(s)|y(s)|\,ds\right]$, where $\lambda \in C(J,R)$, $\int_{t_0}^{\infty}\lambda(t)\,dt < \infty$ and $c_1 > 0$.*

Then every solution $y(t)$ approaches to zero as $t \to \infty$ whenever $x = 0$ of (N) *is EAS.*

PROOF: Since

$$y(t,t_0,y_0) = x(t,t_0,y_0)+\int_{t_0}^{t} \Phi(t,s,y(s))[g(t,y(s))+h(t,y(s),Ty(s)]\,ds$$

we have

$$\begin{aligned}|y(t)| &\le K_1|y_0|e^{-c_1(t-t_0)} \\ &\quad + \int_{t_0}^{t} K_1 e^{-c_1(t-s)}|g(t,y(s)) + h(t,y(s),Ty(s)|\,ds \\ &\le K_1|y_0|e^{-c_1(t-t_0)} + \varepsilon K_1 \int_{t_0}^{t} e^{-c_1(t-s)}|y(s)|\,ds \\ &\quad + K_1 \int_{t_0}^{t} e^{-c_1(t-s)}\lambda(s)\left\{|y(s)| + e^{-c_1 s}\int_{t_0}^{s}\mu(\tau)|y(\tau)|\,d\tau\right\} ds.\end{aligned}$$

Then

$$\begin{aligned}u(t) =&|y(t)|e^{c_1 t} \le K_1|y_0|e^{c_1 t_0} + \varepsilon K_1 \int_{t_0}^{t} e^{c_1 s}|y(s)|\,ds \\ &+ K_1 \int_{t_0}^{t} e^{c_1 s}\lambda(s)|y(s)|\,ds \\ &+ K_1 \int_{t_0}^{t} \lambda(s)\int_{t_0}^{s}\mu(\tau)e^{c_1\tau}|y(\tau)|\,d\tau\,ds \\ =&K_1 u(0) + \varepsilon K_1 \int_{t_0}^{t} u(s)\,ds + K_1 \int_{t_0}^{t}\lambda(s)u(s)\,ds \\ &+ K_1 \int_{t_0}^{t}\lambda(s)\int_{t_0}^{s}\mu(\tau)u(\tau)\,d\tau\,ds.\end{aligned}$$

In view of Lemma 2, we have

$$u(t) \le K_1 u_0 \exp\left\{\int_{t_0}^{t}\left[\varepsilon K_1 + K_1\lambda(s) + K_1\lambda(s)\int_{t_0}^{s}\mu(\tau)\,d\tau\right] ds\right\}.$$

This implies that

$$|u(t) \leq K_1 |y_0| \exp[-c_1(t-t_0)] \times \\ \exp\left\{ \int_{t_0}^{t} \left[\varepsilon K_1 + K_1\lambda(s) + K_1\lambda(s) \int_{t_0}^{s} \mu(\tau)\, d\tau \right] ds \right\}.$$

Therefore the right-hand side of this inequality approaches to zero if K_1 and $|y_0|$ are small enough.

Corresponding to the function $V \in C(R^+ \times R^n, R)$ we define the total derivative V' with respect to (N) by

$$V'_{(N)}(t,x) = \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, x+hf(t,x)) - V(t,x)]$$

and if $x(t)$ is a solution of (N) we denote by $V'(t,x(t))$ the upper right-hand Dini derivative of $V(t,x(t))$, i.e.,

$$V'(t,x(t)) = \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, x(t+h)) - V(t,x)].$$

It is well-known that $V'_{(N)}(t,x) = V'(t,x(t))$ if V is Lipschitzian with respect to x.

Athanassov [**1**] proved Massera type converse theorem for the kind of EASV by constructing a suitable Liapunov function. We prove a converse theorem for GEASV. The technique and results are similar to those in [**1**] and [**5**].

THEOREM 4. *Assume that* $x = 0$ *of* (N) *is GEASV. If* $p'(t)$ *exists and is continuous on* R^+*, then there exists a function* $V \in C(R^+ \times R^n, R)$ *satisfying*

(i) $|x| \leq V(t,x) \leq K(t)|x|$ *for all* $(t,x) \in R^+ \times \mathbf{R}^n$,
(ii) $|V(t,x) - V(t,y)| \leq K(t)|x-y|$ *for all* (t,x), $(t,y) \in R^+ \times \mathbf{R}^n$,
(iii) $V'_{(N)}(t,x) \leq -p'(t)V(t,x)$ *for all* $(t,x) \in R^+ \times \mathbf{R}^n$.

PROOF: We define

$$V(t,x) = \sup_{\tau \geq 0} |x(t+\tau, t, x)| e^{p(t+\tau)-p(t)}.$$

By Lemma 2.2 and the assumption, we have

$$\begin{aligned}|x(t,t_0,x_0)| &\le |x_0| \int_0^1 |\Phi(t,t_0,sx_0)|\,ds \\ &\le |x_0| K(t_0) e^{p(t_0)-p(t)}.\end{aligned}$$

(i).

$$\begin{aligned}|x| = |x(t,t,x)| &\le \sup_{\tau\ge 0} |x(t+\tau,t,x)| e^{p(t+\tau)-p(t)} = V(t,x) \\ &\le K(t)|x|\end{aligned}$$

for all $(t,x) \in R^+ \times R^n$.

(ii). In view of Lemma 2.3, we have

$$\begin{aligned}|V(t,x) - V(t,y)| &\le |x-y| \sup_{\eta\in\mathbf{R}^n} |\Phi(t,t_0,\eta)| e^{p(t+\tau)-p(t)} \\ &\le K(t)|x-y|\end{aligned}$$

for all (t,x), $(t,y) \in R^+ \times R^n$. Thus $V(t,x)$ is Lipschitzian in x of $K(t)$.

From (i) and the uniqueness of solutions of (N), it follows that $V(t,x)$ is defined on $R^+ \times R^n$. We must prove that $V(t,x)$ is continuous.

Let $\delta \ge 0$ and (t,x), $(t,y) \in R^+ \times R^n$. Then

$$\begin{aligned}|V(t+\delta,y) - V(t,x)| \le &|V(t+\delta,y) - V(t+\delta,x)| \\ &+ |V(t+\delta,x) - V(t+\delta,x(t+\delta,t,x))| \\ &+ |V(t+\delta,x(t+\delta,t,x)) - V(t,x)|.\end{aligned}$$

The first two terms on the right-hand side are small when $|y-x|$ and δ are small because $V(t,x)$ is Lipschitzian in x for $K(t)$ and $x(t+\delta,t,x)$ is continuous in δ.

Let us consider the third term. Since

$$x(t+\delta+\tau, t+\delta, x(t+\delta,t,x)) = x(t+\delta+\tau,t,x),$$

we have

$$
\begin{aligned}
&V(t+\delta, x(t+\delta, t, x)) - V(t,x)| \\
&=|\sup_{\tau\geq\delta} |x(t+\tau,t,x)|e^{p(t+\tau)-p(t+\delta)} \\
&\quad - \sup_{\tau\geq 0} |x(t+\tau,t,x)|e^{p(t+\tau)-p(t)}| \\
&=|a(\delta)-a(0)|,
\end{aligned}
$$

where

$$a(\delta) = \sup_{\tau\geq\delta} |x(t+\tau,t,x)|e^{p(t+\tau)-p(t+\delta)}.$$

Since $|x(t+\tau,t,x)|e^{p(t+\tau)-p(t)}$ is bounded continuous for all $\tau \geq 0$ and so the nondecreasing function $a(\delta)$ tends to $a(0)$ as $\delta \to 0$. Hence the continuity of $V(t,x)$ follows.

(iii).

$$
\begin{aligned}
V'_{(N)}(t,x) &= \limsup_{h\to 0+} \frac{1}{h}\Big[\sup_{\tau\geq h} |x(t+\tau,t,x)|e^{p(t+\tau)-p(t+h)} \\
&\quad - \sup_{\tau\geq 0} |x(t+\tau,t,x)|e^{p(t+\tau)-p(t)}\Big] \\
&\leq \limsup_{h\to 0+} \frac{1}{h}\Big[\sup_{\tau\geq 0} |x(t+\tau,t,x)|e^{p(t+\tau)-p(t)}\{e^{p(t)-p(t+h)}-1\}\Big] \\
&\leq -p'(t)V(t,x).
\end{aligned}
$$

This completes the proof.

Finally, we can obtain the GEASV for (P_1) when $x = 0$ of (N) is GEASV by using the following two basic comparison lemmas.

LEMMA 5. *Assume that $V \in C(R^+ \times \mathbf{R}^n, R)$ is Lipschitzian in x with Lipschitz constant L. If $x(s)$ and $y(s)$ are differential functions defined for $s \geq t$ with $x(t) = y(t) = x$, then*

$$V'(t,y(t)) \leq V'(t,x(t)) + L|y'(t) - x'(t)|.$$

PROOF:

$$\begin{aligned} V'(t,y(t)) \le & \limsup_{h\to 0+} \frac{1}{h}[V(t+h,x(t+h)) - V(t,y(t)] \\ & + \limsup_{h\to 0+} \frac{1}{h}[V(t+h,y(t,h)) - V(t+h,x(t+h))] \\ \le & \limsup_{h\to 0+} \frac{1}{h}[V(t+h,x(t+h)) - V(t,x(t))] \\ & + \limsup_{h\to 0+} \frac{1}{h} L|y(t+h) - x(t+h)|. \end{aligned}$$

LEMMA 6 (Conti, [**1**, Lemma 3.1]). *Let $x(t) = x(t,t_0,x_0)$ be a solution of* (N) *existing for $t \ge t_0$. Suppose $V \in C(R^+ \times \mathbf{R}^n, R)$, $V(t,x)$ is Lipschitzian in x and $V'(t,x)$ satisfies for all $(t,x) \in R^+ \times R^n$,*

$$V'(t,x) \le w(t,V(t,x)),$$

where $w \in C(R^+ \times R, R)$. Let $r(t) = r(t,t_0,u_0)$ be the maximal solution of the scalar differential equation

$$u' = w(t,u), \quad u(t_0) = u_0 \ge 0$$

existing for $t \ge t_0$. Then, for $t \ge t_0$,

$$V(t,x(t)) \le r(t),$$

whenever $V(t_0,x_0) \le u_0$.

THEOREM 7. *Let $x = 0$ of* (N) *be GEASV. Assume that in* (P_1) *the perturbing term $g(t,y)$ satisfies*

$$|g(t,y)| \le \varphi(t,|y|), \qquad t \ge t_0 \ge 0, \quad |y| < \infty,$$

where $\varphi \in C(R^+ \times R^+, R^+)$ is increasing in x for $t \in R^+$. If the maximal solution of the scalar differential equation

$$\text{(S)} \qquad u' = [-p'(t) + \lambda(t)]K(t)u, \qquad u(t_0) = u_0 \ge 0,$$

where $p(t)$ and $K(t)$ are the functions from the definition of GEASV, is GEASV, then every solution of (P_1) *is GEASV.*

PROOF: By the assumption, we have

$$|x(t)| \le K(t_0)e^{p(t_0)-p(t)}, \qquad t \ge t_0 \ge 0, \quad |x_0| < \infty$$

for some continuous function $K(t) > 0$ and $p(t) \in \kappa$ with $p(t) \to \infty$ as $t \to \infty$. Then, by Lemma 5,

$$\begin{aligned} V'_{(P_1)}(t,x) &\le V'_{(N)}(t,x) + K(t)|g(t,x)| \\ &\le -p'(t)V(t,x) + K(t)\varphi(t,|x|) \\ &\le -p'(t)K(t)|x| + K(t)\varphi(t,|x|) \end{aligned}$$

for all $t \ge t_0 \ge 0$ and $|x| < \infty$.

Let $r(t) = r(t,t_0,u_0)$ be the maximal solution of (S) existing for $t \ge t_0 \ge 0$ with $|u_0| \le K(t)|y_0|$, $|y_0| < \infty$. Let $y(t)$ be any solution (P_1) existing for $t \ge t_0 \ge 0$. Then $V(t_0,y_0) \le K(t)|y_0| \le u_0$. In view of Lemma 6, we have $V(t,y) \le r(t)$. Thus we have

$$|y(t)| \le V(t,y) \le r(t).$$

Therefore the result follows from the assumption that $r(t)$ is GEASV.

REFERENCES

1. Zh.S. Athanassov, *Perturbation theorems for nonlinear systems of ordinaly differential equations*, J. Math. Anal. Appl. **86** (1982), 194–207.
2. F. Brauer, *Perturbations of nonlinear systems of differential equations II*, J. Math. Anal. Appl. **17** (1967), 418–434.
3. F.M. Dannan and S. Elaydi, *Lipschitz stability of nonlinear systems of differential equations II. Liapunov functions*, J. Math. Anal. Appl. **143** (1989), 517–529.
4. S. Elaydi and H.R. Farran, *Exponentially asymptotically stable dynamical systems*, Appl. Anal. **25** (1987), 243–252.
5. V. Lakshimikantham and S. Leela, *Differential and integral inequalities*, in "Theory and Applications," I, Ordinary Differential Equations, Academic Press, 1969.

6. B.G. Pachpatte, *Stability and asymptotic behavior of perturbed nonlinear system*, J. Differential Equations **16** (1974), 14–25.

Department of Mathematics
Chungnam National University
Taejon, 305-764, Korea
and
Ministry of Education
Seoul, 110-760, Korea