JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 3, June 1990

On the Radon-Nikodym Derivative of a Measure Taking Values in a Dual Banach Space

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ABSTRACT. In this paper, we give the Radon-Nikodym derivative of a measure taking values in a dual Banach space.

Let X be a Banach space with dual X^* and (Ω, Σ, μ) a finite measure space. If $f : \Omega \to X^*$ is bounded and weakly measurable, then it can easily be shown that for every $E \in \Sigma$, there exists $x_E^* \in X^*$ such that for every $x \in X$,

$$x_E^*(x) = \int_E \hat{x} \circ f \, d\mu$$

and for every $E \in \Sigma$, there exists $x_E^{***} \in X^{***}$ such that for every $x^{**} \in X^{**}$,

$$x_E^{***}(x^{**}) = \int_E x^{**} \circ f \, d\mu.$$

The element x_E^* is called the *weak* integral of f over E*, denoted by $(w^*) - \int_E f \, d\mu$, and x_E^{***} is called the *Dunford integral of f over E*, denoted by $(D) - \int_E f \, d\mu$.

In the case that $(D) - \int_E f \, d\mu \in X^*$ for each $E \in \Sigma$, then f is called *Pettis integrable* and we write $(P) - \int_E f \, d\mu$ instead of $(D) - \int_E f \, d\mu$ to denote the *Pettis integral of f over E*.

For a subset A of X^*, we denote the weak* closed convex hull of A in X^* by $\overline{co}^{w^*}(A)$.

We are now able to prove the mean value theorem for the weak* integral.

Received by the editors on 11 June 1990.
1980 *Mathematics subject classifications*: Primary 28B05.

LEMMA 1. *Let $f : \Omega \to X^*$ be weak* integrable with respect to μ. Then for each $E \in \Sigma$ with $\mu(E) > 0$,*

$$\frac{1}{\mu(E)}\left((w^*) - \int_E f\,d\mu\right) \in \overline{co}^{w^*}(f(E)).$$

PROOF: Suppose there is a set $E \in \Sigma$ of positive μ-measure such that $\frac{1}{\mu(E)}\left((w^*) - \int_E f\,d\mu\right) \notin \overline{co}^{w^*}(f(E))$.

By the Hahn-Banach theorem and the fact that $(X^*, \text{weak}^*)^* = X$, we can select $x \in X$ and real α such that

$$\frac{1}{\mu(E)} \int_E \hat{x} \circ f\,d\mu < \alpha \le \hat{x} \circ f(w)$$

for all $w \in E$. Integrating over E yields

$$\int_E \hat{x} \circ f\,d\mu < \alpha\mu(E) \le \int_E \hat{x} \circ f\,d\mu,$$

a contradiction. This completes the proof.

The following theorem was proved by Bator in [**1**].

THEOREM 2. [**1**]. *Let X be a Banach space and (Ω, Σ, μ) a finite measure space. Suppose $f : \Omega \to X^*$ is bounded and weakly measurable. Then f is Pettis integrable if and only if for every $x^{**} \in X^{**}$, there exists a bounded sequence (x_n) in X such that both of the following hold;*

(1) *$\hat{x}_n \circ f$ converges a.e. to $x^{**} \circ f$, and*
(2) *$\hat{x}_n\left((w^*) - \int_E f\,d\mu\right)$ converges to $x^{**}\left((w^*) - \int_E f\,d\mu\right)$ for every $E \in \Sigma$.*

We get the following theorem from Lemma 1 and Theorem 2.

It gives the necessary conditions on the range of a dual Banach space valued function that will guarantee that function to be Pettis integrable.

THEOREM 3. *Let X be a Banach space and (Ω, Σ, μ) a finite measure space. Suppose $f : \Omega \to X^*$ is bounded and weakly measurable. Then f is Pettis integrable if for every $x^{**} \in X^{**}$, there exist a* *bounded sequence* (x_n) *in* X *and a measurable set* A *of measure zero such that*

(1) $\hat{x}_n \circ f$ *converges to* $x^{**} \circ f$ *on* $\Omega - A$, *and*
(2) $f(\Omega - A)$ *is weak* closed and convex.*

PROOF: For $x^{**} \in X^{**}$, let (x_n) be a bounded sequence in X and A be a null set that satisfy the conditions (1) and (2). By Theorem 2, it suffices to show that $\hat{x}_n \left((w^*) - \int_E f\, d\mu\right)$ converges to $x^{**}\left((w^*) - \int_E f\, d\mu\right)$ for every $E \in \Sigma$.

If $\mu(E) = 0$, then $(w^*) - \int_E f\, d\mu = 0$ and the convergence holds trivially.

Let E be a measurable set of positive measure. With the help of Lemma 1, we have

$$\frac{1}{\mu(E)}\left((w^*) - \int_E f\, d\mu\right) = \frac{1}{\mu(E-A)}\left((w^*) - \int_{E-A} f\, d\mu\right)$$
$$\in \overline{co}^{w^*} f(E - A) \subseteq \overline{co}^{w^*} f(\Omega - A).$$

By the condition (2), we have

$$\overline{co}^{w^*} f(\Omega - A) = f(\Omega - A)$$

and

$$\frac{1}{\mu(E)}\left((w^*) - \int_E f\, d\mu\right) \in f(\Omega - A).$$

By the condition (1), $\hat{x}_n \left((w^*) - \int_E f\, d\mu\right)$ converges to

$$x^{**}\left((w^*) - \int_E f\, d\mu\right).$$

LEMMA 4. *Let* $f : \Omega \to X^*$, (Ω, Σ, μ) *a finite measure space and* $\nu : \Sigma \to X^*$ *a* μ*-continuous vector measure.*

Then the set

$$H = \{x \in X : \hat{x} \circ f \in L_1(\mu) \text{ and } \hat{x} \circ \nu(A) = \int_A \hat{x} \circ f\, d\mu \text{ for } A \in \Sigma\}$$

is weak sequentially closed in the subspace X of X^{**}.*

PROOF: From the definition of H, it follows that

$$(1) \qquad \int_A |\hat{x} \circ f| \, d\mu \le \|\hat{x}\| \, \|\nu\|(A) \quad \text{for } A \in \Sigma \text{ and } x \in H,$$

where $\|\nu\|$ denotes the semivariation of ν.

Suppose that $\{x_n\}$ is a sequence in H such that $\hat{x}_n(x^*) \to \hat{x}(x^*)$ for $x^* \in X^*$. By the Uniform Boundedness Theorem, $\sup\{\|\hat{x}_n\| : n = 1, 2, \dots\} < \infty$. Hence, by (1) and the μ-continuity of $\|\nu\|$ $\lim_{\mu(A)\to 0} \int_A |\hat{x}_n \circ f| \, d\mu = 0$ uniformly in $n \in N$. Since $\hat{x}_n \circ f \to \hat{x} \circ f$ on Ω, it follows from Vitali's convergence theorem that $\hat{x} \circ f \in L_1(\mu)$ and

$$\int_A \hat{x} \circ f \, d\mu = \lim_n \int_A \hat{x}_n \circ f \, d\mu = \lim_n \hat{x}_n \circ \nu(A) = \hat{x} \circ \nu(A)$$

for $A \in \Sigma$. This yields $x \in H$.

The next corollary now follows quickly in the same manner as the proof of Lemma 4.

COROLLARY 5. *Let $f : \Omega \to X^*$, (Ω, Σ, μ) a finite measure space and $\nu : \Sigma \to X^*$ a μ-continuous vector measure.*

Then the set

$$K = \{x^{**} \in X^{**} : x^{**} \circ f \in L_1(\mu)$$

and

$$x^{**} \circ \nu(A) = \int_A x^{**} \circ f \, d\mu \text{ for } A \in \Sigma\}$$

is weak sequentially closed.*

With the help of Lemma 4, we get the weak* integrable Radon-Nikodym derivative of a measure $\nu : \Sigma \to X^*$.

THEOREM 6. *Suppose $f : \Omega \to X^*$ is such that $\hat{x} \circ f \in L_1(\mu)$ for all $x \in M$ where M is a weak* sequentially dense subset of X, and $\nu : \Sigma \to X^*$ is a vector measure with*

$$\hat{x} \circ \nu(A) = \int_A \hat{x} \circ f \, d\mu \quad \text{for } A \in \Sigma \text{ and } x \in M.$$

Then f is weak integrable and $\nu(A) = (w^*) - \int_A f\,d\mu$ for $A \in \Sigma$.*

PROOF: Since $\hat{x} \circ \nu(A) = \int_A \hat{x} \circ f\,d\mu$ for $x \in M$ and $A \in \Sigma$, $\hat{x} \circ \nu \ll \mu$ for all $x \in M$. Since M is a weak* sequentially dense subset of X, $\hat{x} \circ \nu \ll \mu$ for all $x \in X$, and hence $\nu \ll \mu$. By Lemma 4, M is weak* sequentially closed and so $M = X$. Hence we have that

$$\hat{x} \circ \nu(A) = \int_A \hat{x} \circ f\,d\mu \quad \text{for every } x \in X.$$

This implies that f is weak* integrable and $\nu(A) = (w^*) - \int_A f\,d\mu$ for $A \in \Sigma$.

With the help of Corollary 5, the next corollary follows in the same manner as the proof of Theorem 6.

COROLLARY 7. *Suppose $f : \Omega \to X^*$ is such that $x^{**} \circ f \in L_1(\mu)$ for all $x^{**} \in M$ where M is a weak* sequentially dense subset of X^{**}, and $\nu : \Sigma \to X^*$ is a vector measure with*

$$x^{**} \circ \nu(A) = \int_A x^{**} \circ f\,d\mu \quad \text{for } A \in \Sigma \text{ and } x^{**} \in M.$$

Then f is Pettis integrable and $\nu(A) = (P) - \int_A f\,d\mu$ for $A \in \Sigma$.

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