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Cohomology Groups of the Separated Spaces

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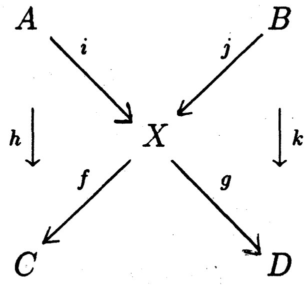
ABSTRACT. In this note we analyse establish the cohomology groups of the topological space which is separated by infinitely many open subspaces.

By a separation (X_1, X_2) of given topological space X, we mean two nonempty open subspaces X_1 and X_2 of X satisfying

$$X_1 \cup X_2 = X \quad \text{and} \quad X_1 \cap X_2 = \emptyset.$$

We say that a space X is separated if a separation (X_1, X_2) of X exists.

We first recall some preliminary lemmas on abelian groups. For this purpose, consider the following diagram of abelian groups and homomorphisms:



LEMMA 1. *Two homomorphisms*

$$\phi : A \oplus B \to X \quad \textit{and} \quad \psi : X \to C \oplus D$$

defined by

$$\begin{aligned} \phi(\xi, \eta) &= i(\xi) + j(\eta), \qquad && (\xi, \eta) \in A \oplus B \\ \psi(x) &= [f(x), g(x)], \qquad && x \in X \end{aligned}$$

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are isomorphisms, provided that the following three conditions are satisfied:

(1) *The two triangles are commutative, i.e.* $h = f \circ i$ *and* $k = g \circ j$,

(2) *The two diagonals are exact, i.e.*

$$\operatorname{Im}(i) = \ker(g), \qquad \operatorname{Im}(j) = \ker(f),$$

(3) *The homomorphisms* h *and* k *are isomorphisms.*

Now let us consider any topological pair (X, A), where X is separated with a given separation (X_1, X_2). Let

$$A_1 = A \cap X_1, \qquad A_2 = A \cap X_2$$

and consider the inclusion maps

$$\begin{aligned} i_1 &: (X_1, A_1) \to (X, A) \\ i_2 &: (X_2, A_2) \to (X, A). \end{aligned}$$

PROPOSITION 2. *For each integer* q, *induced homomorphisms*

$$\begin{aligned} i_1^* &: H^q(X, A) \to H^q(X_1, A_1), \\ i_2^* &: H^q(X, A) \to H^q(X_2, A_2) \end{aligned}$$

are epimorphisms and the map

$$\phi = i_1^* \oplus i_2^* : H^q(X, A) \to H^q(X_1, A_1) \oplus H^q(X_2, A_2)$$

defined by $\phi(x) = (i_1^*(x), i_2^*(x))$ *is an isomorphism.*

PROOF: Let us first consider the following commutative diagram

$$\begin{array}{ccc} & (X_1, A_2) & \\ {\scriptstyle e_1} \swarrow & & \searrow {\scriptstyle f_1} \\ (X_1 \cup A, A) & \xrightarrow[g_1]{\hspace{2cm}} & (X, X_2 \cup A) \end{array}$$

of inclusion maps. The inclusion map e_1 is the excision of the open set A_2 from A as well as $X_1 \cup A$ and the inclusion map f_1 is the excision of the open set X_2 from X and $X_2 \cup A$. Since A_2 and X_2 are also closed in $X_1 \cup A$ and X, respectively, it follows from the Excision Asiom (VI) in [5] that the induced homomorphisms e_1^* and f_1^* are isomorphisms for every integer q.

$$\begin{array}{ccc} & H^q(X_1, A_1) & \\ e_1^* \nearrow & & \nwarrow f_1^* \\ H^q(X_1 \cup A, A) & \xleftarrow[g_1^*]{\qquad\qquad} & H^q(X, X_2 \cup A) \end{array}$$

Since $g_1 \circ e_1 = f_1$, it follows that

$$g_1^* = (e_1^*)^{-1} \circ f_1^*$$

is also an isomorphism.

Similary, the corresponding inclusion maps e_2. f_2 and g_2 induce the isomorphisms e_2^*, f_2^* and g_2^*, respectively, for each p.

$$\begin{array}{ccc} & (X_2, A_2) & \\ e_2 \swarrow & & \searrow f_2 \\ (X_2 \cup A, A) & \xrightarrow[g_2]{\qquad\qquad} & (X, X_1 \cup A) \end{array}$$

$$\Longrightarrow \quad \begin{array}{ccc} & H^q(X_2, A_2) & \\ e_2^* \nearrow & & \nwarrow f_2^* \\ H^q(X_2 \cup A, A) & \xleftarrow[g_2^*]{\qquad\qquad} & H^q(X, X_1 \cup A) \end{array}$$

Next we consider the following commutative diagram of inclusion maps.

$$\begin{array}{ccccc} (X_1 \cup A, A) & & & & (X_2 \cup A, A) \\ & \searrow^{\bar{i}_1} & & \swarrow^{\bar{i}_2} & \\ g_1 \downarrow & & (X, A) & & \downarrow g_2 \\ & \swarrow_{\bar{j}_2} & & \searrow_{\bar{j}_1} & \\ (X, X_2 \cup A) & & & & (X, X_1 \cup A) \end{array}$$

Here, the diagonals are the inclusion maps of two triples

$$(X, X_1 \cup A, A), \qquad (X, X_2 \cup A, A).$$

From the above diagram, we obtain the following diagram of induced homomorphisms;

$$\begin{array}{ccccc}
H^q(X_1 \cup A, A) & & & & H^q(X_2 \cup A, A) \\
 & \nwarrow^{\bar{i}_1^*} & & \nearrow^{\bar{i}_2^*} & \\
g_1^* \uparrow & & H^q(X, A) & & \uparrow g_2^* \\
 & \nearrow^{\bar{j}_1^*} & & \nwarrow^{\bar{j}_2^*} & \\
H^1(X, X_2 \cup A) & & & & H^1(X, X_1 \cup A)
\end{array}$$

Here, the two triangles are commutative, two diagonals are exact and g_1^*, g_2^* are isomorphisms. By the lemma 1, the homomorphism

$$\bar{i}_1^* \oplus \bar{i}_2^* : H^q(X, A) \to H^q(X_1 \cup A, A) \oplus H^q(X_2 \cup A, A)$$

is an isomorphism.

Finally let us consider the following commutative diagram of inclusion maps:

$$\begin{array}{ccccc}
(X_1, A_1) & & & & (X_2, A_2) \\
 & \searrow^{i_1} & & \swarrow^{i_2} & \\
e_1 \downarrow & & (X, A) & & \downarrow e_2 \\
 & \nearrow^{\bar{i}_1} & & \nwarrow^{\bar{i}_2} & \\
(X_1 \cup A, A) & & & & (X_2 \cup A, A)
\end{array}$$

From the above diagram, we obtain the following diagram of abelian groups and induced homomorphisms

$$\begin{array}{ccccc}
H^q(X_1, A_1) & & & & H^q(X_2, A_2) \\
 & \nwarrow^{i_1^*} & & \nearrow^{i_2^*} & \\
e_1^* \uparrow & & H^q(X, A) & & \uparrow e_2^* \\
 & \swarrow^{\bar{i}_1^*} & & \searrow^{\bar{i}_2^*} & \\
H^q(X_1 \cup A, A) & & & & H^q(X_2 \cup A, A)
\end{array}$$

Since two triangles are commutative and e_1^*, e_2^* are isomorphisms, it follows that the homomorphism

$$\begin{aligned} i_1^* \oplus i_2^* &= (e_1^* \circ \bar{i}_1^*) \oplus (e_2^* \circ \bar{i}_2^*) \\ &= (e_1^* \oplus e_2^*) \circ (\bar{i}_1^* \oplus \bar{i}_2^*) \end{aligned}$$

is an isomorphism and i_1^* and i_2^* are epimorphisms. Hence

$$H^q(X, A) \cong H^q(X_1, A_1) \oplus H^q(X_2, A_2).$$

This completes the proof.

Now we will analyse the cohomology groups of the topological space which is separated by infinitely many open subspaces.

THEOREM 3. *Let $\{X_\alpha \mid X_\alpha$ is a topological space, $\alpha \in J\}$ be an indexed family of disjoint topological spaces. Let A_α be a subspace of X_α for each α, and let $X = \sum_{\alpha\in J} X_\alpha$ and $A = \sum_{\alpha \in J} A_\alpha$ be the topological sums. Then we have $H^q(X, A) \cong \bigoplus_{\alpha\in J} H^q(X_\alpha, A_\alpha)$.*

PROOF: We shall order the index set J into a well ordered set

$$J = \{\alpha_0, \alpha_1, \dots, \alpha_n, \dots, \alpha_w, \dots\}$$

and let $J' = \{\beta_0, \beta_1, \dots\}$ be any other well ordered set. Then

$$\Omega = J \cup J' = \{\alpha_0, \alpha_1, \dots, \beta_0, \beta_1, \dots\}$$

is also a well ordered set. Now we let

$$\{X_{\beta_0}, X_{\beta_1}, \dots\}$$

be the collection of disjoint topological spaces such that $X_{\alpha_i} \cap X_{\beta_j} = \emptyset$ and $A_{\beta_j} \subset X_{\beta_j}$ for each $\alpha_j \in J$ and $\beta_j \in J'$. For each $\lambda \in \Omega$, we denote $Y_\lambda = \sum_{\alpha<\lambda} X_\alpha$, $B_\lambda = \sum_{\alpha<\lambda} A_\alpha$ which are the topological sums of the spaces X_α and A_α, respectively. and we will prove the theorem by transfinite induction, that is, for each $\lambda \in \Omega$ we have $H^q(Y_\lambda, B_\lambda) \cong \bigoplus_{\alpha<\lambda} H^q(X_\alpha, A_\alpha)$. Since $Y_{\alpha_0} = \emptyset$, $B_{\alpha_0} = \emptyset$, $Y_{\alpha_1} = X_{\alpha_0}$ and $B_{\alpha_1} = A_{\alpha_0}$, we have $H^q(Y_{\alpha_0}, B_{\alpha_0}) = \{0\}$ and $H^q(Y_{\alpha_1}, B_{\alpha_1}) = H^q(X_{\alpha_0}, A_{\alpha_0}) \cong \bigoplus_{\alpha<\alpha_1} H^q(X_\alpha, A_\alpha)$.

Now we suppose that the induction hypothesis holds, that is, for each $\lambda \in \Omega$ and each q, $H^q(Y_\mu, B_\mu) \cong \bigoplus_{r<\mu} H^q(X_r, A_r)$ holds for all $\mu \le \lambda$. Let $\lambda' = \lambda + 1$ and we will show that

$$H^q(Y_{\lambda'}, B_{\lambda'}) \cong \bigoplus_{r<\lambda'} H^q(X_r, A_r).$$

Since $Y_{\lambda'} = Y_\lambda \cup x_\lambda$ is a topological sum of Y_λ and X_λ, $B_{\lambda'} \cap Y_\lambda = B_\lambda$ and $B_{\lambda'} \cap X_\lambda = A_\lambda$, we have

$$\begin{aligned} H^q(Y_{\lambda'}, B_{\lambda'}) &\cong H^q(Y_\lambda, B_\lambda) \oplus H^q(X_\lambda, A_\lambda) \\ &\cong \left[\bigoplus_{r<\lambda} H^q(X_r, A_r)\right] \oplus H^q(X_\lambda, A_\lambda) \\ &= \bigoplus_{r<\lambda'} H^q(X_r, A_r). \end{aligned}$$

By the transfinite induction, our assertion holds for each ordinal in Ω. Hence we get

$$H^q(X, A) = H^q(Y_{\beta_0}, B_{\beta_0}) \cong \bigoplus_{r<\beta_0} H^q(X_r, A_r) = \bigoplus_{\alpha \in J} H^q(X_\alpha, B_\alpha)$$

for each integer q.

In the case of $A_\alpha = \emptyset$ for all α in J, we have

$$H^q(X) \cong \bigoplus_{\alpha \in J} H^q(X_\alpha).$$

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