JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 3, June 1990

Zeeman's Tolerance Stability Conjecture and Takens' Conjecture

JONG MYUNG KIM

ABSTRACT. We do an attempt to solve the Zeeman's tolerance stability conjecture and Takens' conjecture using the concepts of chains.

Let X be a compact metric space with a metric d. Let $H(X)$ be the set of all homeomorphisms of X to itself: the topology on $H(X)$ is induced by the metric

$$d_0(f,g) = \sup\{d(f(x),g(x)) : x \in X\}.$$

Let $K(X)$ be the set of all nonempty compact subspaces of X with the Hausdorff metric : for any nonempty compact subsets A, B of X,

$$\rho(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(a,B) = \inf\{d(a,b) : b \in B\}$. Then $K(X)$ is again a compact metric space. Let $K(K(X))$ be the set of all nonempty compact subspaces of $K(X)$ with the Hausdorff metric $\bar{\rho}$.

For any $f \in H(X)$ and $x \in X$, the set

$$O(f,x) = \overline{\{f^n(x) : n \in Z\}}$$

is called the *f-orbit through x*. Since the set $O(f,x)$ can be interpreted as a point in K(X), we can consider the closure of the set$\{O(f,x) : x \in X\}$ in $K(X)$, which is denoted by $O(f)$. The set $O(f)$ also may be interpreted as a point of $K(K(X))$. Hence we consider the orbit map

$$O : H(X) \to K(K(X))$$

Received by the editors on May 26, 1990.
1980 *Mathematics subject classifications*: Primary 58F15, Secondary 54H20.

sending $f \in H(X)$ to $O(f)$.

We say that $f \in H(X)$ is *tolerance stable* if the map $O : H(X) \to K(K(X))$ is continuous in f. The Zeeman's tolerance stability conjecture is the following : there is a residual subset S of $H(X)$ such that each $f \in H(X)$ is tolerance stable.

As we can see in [**6**], this conjecture does not hold. In [**6**], Taken claimed that if the definition of orbit is changed somewhat, the conjecture is true. The concept of orbit which is used in [**6**] is the notion of extended orbit.

First we have a question under which conditions the map $O : H(X) \to K(K(X))$ is continuous. To find a condition, we introduce the concept of persistence. $f \in H(X)$ is called *persistent* if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $d_0(f,g) < \delta$ and $x \in X$, then there are y, $z \in X$ satisfying

$$d(f^n(x), g^n(y)) < \varepsilon \quad \text{and} \quad d(f^n(z), g^n(x)) < \varepsilon,$$

for all $n \in \mathbf{Z}$.

THEOREM 1. *The orbit map $O : H(X) \to K(K(X))$ is continuous at $f \in H(X)$ if f is persistent.*

PROOF: Let $\varepsilon > 0$ be arbitrary. Since f is persistent, we can choose $\delta > 0$ such that if $d_0(f,g) < \delta$ and $x \in X$, then

$$d(f^n(x), g^n(y)) < \frac{\varepsilon}{3} \quad \text{and} \quad d(f^n(z), g^n(x)) < \frac{\varepsilon}{3}$$

for some y, $z \in X$ and all $n \in \mathbf{Z}$. The proof is completed by showing that $O(g) \subset B_\varepsilon(O(f))$ and $O(f) \subset B_\varepsilon(O(g))$, where $B_\varepsilon(\cdot)$ denotes the ε-neighborhood of $(\cdot)$. Let $A \in O(g)$. Then there exists $x \in X$ such that $\rho(A, O(g,x)) < \frac{\varepsilon}{2}$. Given $x \in X$, we select $z \in X$ satisfying

$$d(f^n(z), g^n(x)) < \frac{\varepsilon}{3}$$

for all $n \in \mathbf{Z}$. Then we have $\rho(O(f,z), O(g,x)) < \frac{\varepsilon}{2}$, and so $\rho(A, O(f,z)) < \varepsilon$. This means that $O(g) \subset B_\varepsilon(O(f))$, and hence O is upper semicontinuous. Let $A \in O(f)$. Then there exists $x \in X$ such that $\rho(A, O(f,x)) < \frac{\varepsilon}{2}$. Given $x \in X$, we can choose $y \in X$ satisfying

$$d(f^n(x), g^n(y)) < \frac{\varepsilon}{3}$$

for all $n \in \mathbf{Z}$. Then we have $\rho(O(f,x), O(g,y)) < \frac{\varepsilon}{2}$, and so $\rho(A, O(g,y)) < \varepsilon$. This means that O is lower semi-continuous. Hence the map O is continuous.

A closed subset $A \subset X$ is called an *ε-orbit* of f, $\varepsilon > 0$, if there exists a sequence $\{x_n\}$ of points of X such that $d(f(x_n), x_{n+1}) < \varepsilon$ and such that $\{x_n\}$ is dense in A. A closed set $A \subset X$ is called an *extended f-orbit* if for any $\varepsilon > 0$ and $\delta > 0$, there is an ε-orbit A_ε of f such that $\rho(A, A_\varepsilon) < \delta$. (See [6]). Let $E_f \subset K(X)$ be the set of all extended f-orbits. Using the concept of extended f-orbit, Takens showed that there is a residual subset S of $H(X)$ such that $E : S \to K(K(X))$, sending $f \in S$ to E_f, is continuous. Moreover, he suggested two conjectures :

Conjecture 1. Let f be a diffeomorphism on a compact (metric) differentiable manifold X. If f satisfies Axiom A and the strong transversality condition, then $E_f = O(f)$.

Conjecture 2. Let X be a compact (metric) differentiable manifold and Diff$(X) \subset H(X)$ is the set of C^1-diffeomorphisms with the C^1-topology. Then there is a residual subset S of Diff(X) such that $E_f = O(f)$ for $f \in S$.

In 1980, K. Sawada gave an affirmative answer for the conjecture 1. (see [4]).

Now we do an attempt to solve the Zeeman's conjecture and Takens' conjecture. For this, we define a partial order relation "$<$" on X, induced by $f \in H(X)$, as follows : for any x, $y \in X$, $x < y$ if and only if there exists an ε-chain $\{x_0, x_1, \ldots, x_n\}$ from x to y, for any $\varepsilon > 0$, where $\{x_0, x_1, \ldots, x_n\}$ is called an ε-chain from x to y if $x_0 = x$, $x_n = y$ and $d(f(x_i), x_{i+1}) < \varepsilon$ for $i = 0, 1, \ldots, n-1$.

LEMMA 2. *The set $P = \{(x,y) \in X \times X : x < y\}$ is compact in $X \times X$.*

PROOF: Let (a,b) be any point in $\overline{P}$. Then we must show that $(a,b) \in P$. Let $\varepsilon > 0$ be arbitrary. Since X is compact, we can choose $0 < \delta < \frac{\varepsilon}{2}$ such that if $d(x,y) < \delta$ then $d(f(x), f(y)) < \frac{\varepsilon}{2}$ for $x, y \in X$. Given $\delta > 0$, there exists $(c,d) \in P$ satisfying $\bar{d}((a,b),(c,d)) < \delta$, where $\bar{d}$ denotes the metric on $X \times X$ induced by d. Since $(c,d) \in P$, we can select an δ-chain

$$\{x_0 = c, x_1, \ldots, x_{n-1}, x_n = d\}$$

from C to d. Then the sequence

$$\{a, x_1, \dots, x_{n-1}, b\}$$

is an ε-chain from a to b. In fact, we have

$$\begin{aligned} d(f(a), x_1) &\leq d(f(a), f(x_0)) + d(f(x_0), x_1) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and

$$\begin{aligned} d(f(x_{n-1}), b) &\leq d(f(x_{n-1}), d) + d(d, b) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This means that $(a, b) \in P$, and so the proof is completed.

We say that the set $\{y \in X : x < y \text{ or } x > y \text{ or } x = y\}$ is an *f-chain orbit* through $x \in X$, which is denoted by $C(f, x)$.

LEMMA 3. *The set $C(f, x)$ is compact in X for each $x \in X$.*

PROOF: Let $A = \{y \in X : x < y\}$ and $B = \{y \in X : x > y\}$. Let $\pi_1 : X \times X \to X$ be the first projection map. Then the set $(\pi_1|_P)^{-1}\{x\}$ is compact in P, and so compact in $X \times X$ by Lemma 2. Since $(\pi_1|_P)^{-1}\{x\} = \{x\} \times A$, the set A is compact in X. Similarly we can show that the set B is compact in X. Since $C(f, x) = A \cup B \cup \{x\}$, $C(f, x)$ is compact in X.

In our theorem, the chain orbits take the place of the orbits in Zeeman's original conjecture. Since every chain orbit in X is compact in X, it may be interpleted as a point in $K(X)$. Thus we define $C(f) \subset K(X)$ to be the set of all chain orbits in X, i.e., $C(f) = \{C(f, x) : x \in X\}$.

LEMMA 4. *The set $C(f)$ is compact in $K(X)$.*

PROOF: Let A be any point in $\overline{C(f)}$. Then we must show that $A \in C(f)$. For any positive integer n, we choose $x_n \in X$ such that $\rho(C(f, x_n), A) < \frac{1}{n}$. Then we may assume that the sequence $\{x_n\}$

converges to $x \in X$. Thus we have $\rho(A, C(f,x)) = 0$. This means that $A = C(f,x)$ and so $A \in C(f)$.

Now we consider the chain orbit map

$$C : H(X) \to K(K(X))$$

sending f to $C(f)$. By Lemma 4, the map C is well-defined. The Zeeman's tolerance stability conjecture does hold if we use the chain orbit map.

LEMMA 5. *Suppose X and Y are metric spaces, with Y compact. Let $h : X \to K(Y)$ be either upper or lower semicontinuous. Then the set of continuity points of h is a residual subset of X.*

THEOREM 6. *There is a residual subset S of $H(X)$ such that the map $C : S \to K(K(X))$ is continuous in each point of S.*

PROOF: By Lemma 5, we complete the proof by showing that the map $C : H(X) \to K(K(X))$ is upper semicontinuous. Choose $f \in H(X)$ and $x \in X$. Let $C(f,x,\delta)$ be the closure of the set

$$\{y \in X : \exists\delta\text{-chain from } x \text{ to } y, \text{ or } \exists\delta\text{-chain from } y \text{ to } x, \text{ or } x = y\}$$

for any $\delta > 0$, and let $C(f,\delta)$ be the closure of the set $\{C(f,x,\delta) : x \in X\}$ of $K(X)$. Then we have $C(f) = \bigcap_{\delta>0} C(f,\delta)$. Since $K(X)$ is compact and $C(f)$ is also compact in $K(X)$, we can choose $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$,

$$C(f,\delta) \subset B_\varepsilon(C(f)),$$

where $B_\varepsilon(\cdot)$ denotes the ε-neighborhood of $(\cdot)$ under the metric ρ. If $d_0(f,g) < \frac{1}{2}\delta_0$ and $0 < \delta < \frac{\delta_0}{2}$, then we have

$$C(g) \subset C(g,\delta) \subset C(f,\delta_0) \subset B_\varepsilon(C(f)).$$

In fact, let $\{x_0, x_1, \ldots, x_n\}$ be an δ-chain from x to y for g, x, $y \in X$. Then it is an δ_0-chain from x to y for f, by the following property :

$$\begin{aligned} d(f(x_i), x_{i+1}) &\leq d(f(x_i), g(x_i)) + d(g(x_i), x_{i+1}) \\ &< \frac{1}{2}\delta_0 + \delta < \delta_0, \end{aligned}$$

for $i = 0, 1, \ldots, n-1$. This means that $C(g, \delta) \subset C(f, \delta_0)$. Consequently, we have shown that for any $\varepsilon > 0$ there exists $\delta_0 > 0$ such that if $d_0(f, g) < \delta_0$ than $C(g) \subset B_\varepsilon(C(f))$, and so the map C is upper semicontinuous.

Finally, we should investigate what the difference is between $O(f)$ and $C(f)$. For this problem, K. Sawada [4] proved the Takens' following conjecture : Let f be a diffeomorphism on a compact differentiable manifold M. If f satisfies Axiom A and the strong transversality condition, then $E_f = O(f)$.

Here we give a necessary condition to be $C(f) = O(f)$. For these object, we need a lemma due to Z. Nitecki and M. Shub [2].

LEMMA 7. *Let X be a compact manifold of* $\dim \geq 2$ *with the metric d, and let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that if $\{(x_i, y_i) \in X \times X : i = 1, \ldots, n\}$ is a finite set of points in $X \times X$ satisfying :*

i) *for each $i = 1, \ldots, n$, $d(x_i, y_i) < \delta_i$ and*
ii) *if $i \neq j$, then $x_i \neq x_j$ and $y_i \neq y_j$;*

then there exists $h \in H(X)$ with $d_0(h, 1_\times) < \varepsilon$ and $h(x_i) = y_i$ for $i = 1, \ldots, n$.

THEOREM 8. *Let X be a compact manifold and $f \in H(X)$ be persistent. Then we have $C(f) = O(f)$.*

PROOF: If X is one-dimensional, then the proof is obvious. Hence we may assume that the dimension of X is larger than 1. By definition, it is clear that $O(f) \subset C(f)$. Thus we must show that $C(f) \subset O(f)$. Let $x \in X$. Then it is enough to show that $C(f, x) \subset O(f, x)$. Let $y \in C(f, x)$. Then we have : $x < y$, or $x > y$, or $x = y$. Suppose that $x < y$, and let $k > 0$ be a positive integer. Since f is persistent, given $\frac{1}{k} > 0$, there exists $\delta_1(k) > 0$ such that if $d_0(f, g) < \delta_1$, then $d(f^n(z), g^n(x)) < \frac{1}{k}$ and $d(f^n(x), g^n(w)) < \frac{1}{k}$ for some z, $w \in X$ and all $n \in \mathbf{Z}$. Given $\delta_1 > 0$, we choose $\delta_2(k) > 0$ satisfying the results of Lemma 7. Let $\{x_0, x_1, \ldots, x_{m_k}\}$ be on δ_2-chain for f from x to y. Then the set $\{(f(x_0), x_1), \ldots, (f(x_{m_k - 1}), x_{m_k})\}$ satisfies the hypothesis of Lemma 7. Hence there exists $h \in H(X)$ such that

$$d_0(h, 1_\times) < \delta_1 \quad \text{and} \quad h(f(x_i)) = x_{i+1}$$

for $i = 0, 1, \ldots, m_k - 1$. By letting $g = h \circ f$, we get $d_0(f, g) < \delta_1$. Thus we can choose $z_k \in X$ satisfying

$$d(f^n(z_k), g^n(x)) < \frac{1}{k},$$

for all $n \in \mathbf{Z}$. In particular, we have

$$d(z_k, x) < \frac{1}{k} \quad \text{and} \quad d(f^{m_k}(z_k), y) < \frac{1}{k}.$$

This means that for any $\varepsilon > 0$, $\overline{B_\varepsilon(y)} \cap O(f, x) \neq \emptyset$. Suppose not. Then there exists $\varepsilon > 0$ such that

$$\overline{B_\varepsilon(y)} \cap \overline{\{f^n(x) : n \in \mathbf{Z}\}} = \emptyset, \quad \text{and so}$$
$$B_\varepsilon(y) \cap \{f^n(z_k) : n \in \mathbf{Z}\} = \emptyset,$$

for some $z_k \in B_\varepsilon(x)$. This contradicts to the fact that for any $k > 0$ there exists $z_k \in X$ satisfying $d(f^{m_k}(z_k), y) < \frac{1}{k}$ for some $m_k \in \mathbf{Z}$. Thus we have $\overline{B_\varepsilon(y)} \cap O(f, x) \neq \emptyset$ for any $\varepsilon > 0$. This implies that $B_\varepsilon(y) \cap O(f, x) \neq \emptyset$ for any $\varepsilon > 0$, and so $y \in O(f, x)$. By now we have shown that if $x < y$ then $y \in O(f, x)$. Similarly we can show that if $x > y$ then $y \in O(f, x)$. This completes the proof.

REMARK 9: In order to get a precise idea of what the relation is between Theorem 8 and Takens' conjecture 1, we should investigate what the difference is between persistent diffeomorphism and Axiom A diffeomorphism with strong transversality condition.

REFERENCES

1. N.P. Bhatia and G.P. Szego, "Stability theory of dynamical systems," Springer-Verlag, New-York, 1970.
2. Z. Nitecki and M. Shub, *Filtrations, decompositions and explosions*, Amer. J. Math. **97** (1976), 1029–1047.
3. J. Ombach, *Consequences of the pseudo orbit tracing property and expansiveness*, J. Austral. Math. Soc. **43** (1987), 301–313.
4. K. Sawada, *Extended f-orbits are approximated by orbits*, Nagoya Math. J. **79** (1980), 33-45.

5. F. Takens, "On Zeeman's tolerance stability conjecture," Lecture Notes in Math. 197, Springer, 1971.
6. ———, "Tolerance Stability, Dynamical Systems-Warwick," Lecture Notes in Math. 468, Springer, 1974, pp. 293–304.

Department of Mathematics Education
Kwandong University
Kangreung, 210-701, Korea