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Second Order Derivations on $C^n[0,1]$

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ABSTRACT. Let $D' : C^n[0,1] \longrightarrow M$ be a second order derivation from the Banach algebra of n times continuously differentiable functions on $[0,1]$ into a Banach $C^n[0,1]$-module M and let D be the primitive of D'. If D' is continuous and $D'(z)$ lies in the 1-differential subspace, then it is completely determined by $D(z)$ and $D'(z)$ where $z(t) = t, 0 \leq t \leq 1$.

1. Introduction

Let $C^n(I)$, $I = [0,1]$ denote the algebra of all complex valued functions on I which have n continuous derivatives. It is well known that $C^n(I)$ is a Banach algebra under the norm

$$\|f\|_n = \max_{t \in I} \sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!}$$

whose structure space is I. A Banach $C^n(I)$-module is a Banach space M together with a continuous homomorphism $\rho : C^n(I) \longrightarrow B(M)$. A derivation, or a module derivation of $C^n(I)$ into M is a linear map $D : C^n(I) \longrightarrow M$ which satisfies the identity

$$D(fg) = \rho(f)D(g) + \rho(g)D(f), f, g \in C^n(I).$$

The continuity ideal for a derivation $D : C^n(I) \longrightarrow M$ is

$$\Im(D) = \{f \in C^n(I) \mid \rho(f)D(\cdot) \text{ is continuous}\}.$$

We use the notation

$$M_{n,k}(t_0) = \{f \in C^n(I) \mid f^{(j)}(t_0) = 0; j = 0, 1, \ldots, k\}.$$

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These are precisely the closed ideals of finite codimension contained in the maximal ideal $M_{n,0}(t_0)$ of functions vanishing at t_0. In 1974 Bade and Curtis [1] proved that there exists a finite set $F = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ in I such that

$$\cap_{i=1}^{m} M_{n,n}(\lambda_i) \subset \Im(D) \subset \cap_{i=1}^{m} M_{n,0}(\lambda_i).$$

The hull F of $\Im(D)$ is called the singularity set for D. If $D : C^n(I) \longrightarrow M$ is a derivation, we have

$$D(p(z)) = \rho(p'(z))D(z), p \in P,$$

where P is the dense subalgebra of polynomials in z. If D is continuous, it is completely determined by this formula. Thus a continuous derivation D is uniquely determined by the vector $D(z)$[2]. We need to define the notion of the k-differential subspace of a Banach $C^n(I)$-module, a concept first introduced by Kantorovitz who named it "semisimplicity manifold" [3, 4]. Let M be a Banach $C^n(I)$-module. The k-differential subspace is the W_k of all vectors m such that the map $p \longrightarrow \rho(p')m$ is continuous on P for the $C^{n-k+1}(I)$ norm. We quote the following results from [5]. If $D(z) \in W_1$, then $D = E + F$ where E is continuous and F is a singular derivation (i.e $F(z) = 0$).

THEOREM 1.1. *$p \longrightarrow \rho(p^{(i)})m$ is continuous for the $C^n(I)$ norm on P if and only if $p \longrightarrow \rho(p^{(i+j)})m$ is continuous for the $C^{n+j}(I)$ norm on P, $(i, j = 0, 1, 2, \ldots)$.*

THEOREM 1.2. *Let M be a $C^n(I)$-module with k-differential subspace W_k. For $m \in W_k$ $(0 \le k \le n)$, we define $|||m|||_k = \sup\{\|\rho(p)m\| \mid \|p\|_{n-k} \le 1\}$. Then*

(1) $\|m\| \le |||m|||_0 \le |||m|||_1 \le \cdots \le |||m|||_k$, *for* $m \in W_k$,
(2) W_k *is a Banach space under the norm* $|||\cdot|||_k$,
(3) W_k *is a $C^n(I)$-module and there exists a unique continuous homomorphism*

$$\gamma_k : C^{n-k}(I) \longrightarrow B(W_k)$$

such that $\gamma_k(p)m = \rho(p)m$, $m \in W_k$, $p \in P$,
(4) *If* $S \in B(W_k)$ *and* $S\rho(z) = \rho(z)S$ *on* W_i, *for some* i $(0 \le i \le k)$, *then* $SW_k \subset W_k$ *and* $|||S|||_k \le \|S\|$, *where* $|||S|||_k$ *is norm of* S *in* $B(W_k)$.

DEFINITION 1.3: Let S is a linear map of $C^n(I)$ into the Banach space M. S is separable if there are maps

$$\Phi_1 : C^n(I) \longrightarrow [0,\infty) \text{ and } \Phi_2 : C^n(I) \longrightarrow [0,\infty)$$

such that

$$\|S(fg)\| \leq \Phi_1(f)\Phi_2(g)$$

for every $f, g \in C^n(I)$.

THEOREM 1.4. (cf. [**6**, Lemma 1.1]) *Let $S : C^n(I) \longrightarrow M$ is a separable, let $X_1 \subset C^n(I)$ be a closed subspace on which S is continuous and suppose $\{f_n\}$, $\{g_n\} \subset C^n(I)$ are sequence such that $f_n g_n \in X_1$ whenever $n \neq m$. Then there is a constant $C > 0$ such that*

$$\|S(f_n g_n)\| \leq C\|f_n\| \, \|g_n\|.$$

2. Second order derivations

DEFINITION 2.1: Let M be a Banach $C^n(I)$-module. A linear map $D' : C^n(I) \longrightarrow M$ is a second order derivation if there exists a derivation $D : C^n(I) \longrightarrow M$ such that $D(C^n(I)) \subset W_1$ and

$$\begin{aligned} D'(fg) =& \rho(f)D'(g) + \gamma_1(f')D(g) \\ &+ \gamma_1(g')D(f) + \rho(g)D'(f), \qquad f, g \in C^n(I), \end{aligned}$$

where $\gamma_1 : C^{n-1}(I) \longrightarrow B(W_1)$ is a unique continuous homomorphism such that $\gamma_1(p)m = \rho(p)m$, $m \in W_1$. We call D primitive of D'.

THEOREM 2.2. *Let $D' : C^n(I) \longrightarrow M$ be a second order derivation. Then its primitive is unique.*

PROOF: Suppose

$$\gamma_1(f')D_1(g) + \gamma_1(g')D_1(f) = \gamma_1(f')D_2(g) + \gamma_1(g')D_2(f),$$

for f, $g \in C^n(I)$. Then $D_1(z) = D_2(z)$. Since

$$\gamma_1(f')D_1(z) + D_1(f) = \gamma_1(f')D_2(z) + D_2(f)$$

$D_1(f) = D_2(f)$, $f \in C^n(I)$.

DEFINITION 2.3: Let $D' : C^n(I) \longrightarrow M$ be a second order derivation. The continuity ideal of D' is defined as

$$\Im(D') = \{f \in C^n(I) \mid \rho(f)D'(\cdot) \text{ is continuous on } C^n(I)\}.$$

THEOREM 2.4. *If $D' : C^n(I) \longrightarrow M$ is a continuous second order derivation with primitive D, then $D : C^n(I) \longrightarrow M$ is continuous.*

PROOF: If $f_n \longrightarrow 0$ in $C^n(I)$, then

$$zf_n \longrightarrow 0 \text{ in } C^n(I).$$

$$\begin{aligned} D'(zf_n) = \rho(z)D'(f_n) + \gamma_1(f_n')D(z) + D(f_n) \\ + \rho(f_n)D'(z) \longrightarrow 0, \text{ as } n \longrightarrow \infty. \end{aligned}$$

Since $f_n' \longrightarrow 0$ in $C^{n-1}(I)$, $D(f_n) \longrightarrow 0$.

COROLLARY 2.5. *Let $D' : C^n(I) \longrightarrow M$ be a second order derivation with primitive D. Then*

$$\Im(D') \subset \Im(D).$$

COROLLARY 2.6. *Let $D' : C^n(I) \longrightarrow M$ be a second order derivation with primitive D. Then hull$(\Im(D)) \subset$ hull$(\Im(D'))$.*

LEMMA 2.7. *If $D : C^n(I) \longrightarrow M$ is a second order derivation with primitive D, then*

$$D'(p) = \rho(p')D'(z) + \gamma_1(p'')D(z), \; p \in P.$$

PROOF: $D'(z^2) = \rho(2z)D'(z) + 2D(z)$. Suppose

$$D'(z^n) = \rho(nz^{n-1})D'(z) + \gamma_1(n(n-1)z^{n-2})D(z),$$

for some n. Then

$$\begin{aligned} D'(z^{n+1}) &= D'(zz^n) \\ &= \rho(z)D'(z^n) + D(z^n) + \rho(nz^{n-1})D(z) + \rho(z^n)D'(z) \\ &= \rho((n+1)z^n)D'(z) + \gamma_1((n+1)nz^{n-1})D(z). \end{aligned}$$

THEOREM 2.8. *Let $D' : C^n(I) \longrightarrow M$ be a continuous second order derivation with primitive D. If $D'(z) \in W_1$, then $D(z) \in W_2$ and*

$$D'(f) = \gamma_1(f')D'(z) + \gamma_2(f'')D(z), \quad f \in C^n(I).$$

PROOF: Since $D'(p) = \rho(p')D'(z) + \gamma_1(p'')D(z)$, $p \in P$,

$$p \longrightarrow \rho(p')D'(z)$$

is continuous in $C^n(I)$. So $p \longrightarrow \gamma_1(p'')D(z)$ is continuous in $C^n(I)$. By Theorem 1.1,

$$D(z) \in W_2.$$

If $p_k \longrightarrow f$ in $C^n(I)$, $p_k \in P$, then $p'_k \longrightarrow f'$ in $C^{n-1}(I)$ and $p''_k \longrightarrow f''$ in $C^{n-2}(I)$. By Theorem 1.2, there exists a unique continuous homomorphism

$$\rho(p)m = \gamma_1(p)m = \gamma_2(p)m, \quad m \in W_2.$$

Hence

$$D'(f) = \lim_{n\to\infty} D'(p_k) = \gamma_1(f')D'(z) + \gamma_2(f'')D(z).$$

COROLLARY 2.9. *Let $D' : C^n(I) \longrightarrow M$ be a continuous second order derivation with primitive D. If $D(z) \in W_2$, then $D'(z) \in W_1$.*

3. The structure of second order derivations

A nontrivial second order derivation $D' : C^n(I) \longrightarrow M$ will be called singular if D vanishes on P (equivalently $D'(z) = 0$).

THEOREM 3.1. *Let $D' : C^n(I) \longrightarrow M$ be a second order derivation. If $D'(z) \in W_1$ and $D(z) \in W_2$, then*

$$D' = E' + F'$$

where E' is continuous and F' is a singular second order derivation.

PROOF: We define $E'(f) = \gamma_1(f')D'(z) + \gamma_2(f'')D(z)$, $f \in C^n(I)$. Since $D(z) \in W_2$,

$$D = E + F$$

where E is continuous and F is singular. By Theorem 1.2, for $f, g \in C^n(I)$,

$$\begin{aligned}
E'(fg) &= \gamma_1(f'g + fg')D'(z) + \gamma_2(f''g + 2f'g' + fg'')D(z) \\
&= \gamma_1(f')\gamma_1(g)D'(z) + \gamma_1(f)\gamma_1(g')D'(z) + \gamma_2(f'')\gamma_2(g)D(z) \\
&+ \gamma_2(f)\gamma(g'')D(z) + 2\gamma_2(f')\gamma_2(g')D(z) \\
&= \gamma_1(g)\{\gamma_1(f')D'(z) + \gamma_2(f'')D(z)\} \\
&+ \gamma_1(f)\{\gamma_1(g')D'(z) + \gamma_2(g'')D(z)\} + \gamma_1(g')E(f) + \gamma_1(f')E(g) \\
&= \rho(f)E'(g) + \gamma_1(f')E(g) + \gamma_1(g')E(f) + \rho(g)E'(f).
\end{aligned}$$

So $E' : C^n(I) \longrightarrow M$ is a continuous second order derivation with primitive E. Then $F' = D' - E'$ is singular.

THEOREM 3.2. *Let $D' : C^n(I) \longrightarrow M$ is a second order derivation with primitive D. If $D' = E' + F'$ where E' is continuous, F' is singular and $D'(z) \in W_1$, then $D(z) \in W_2$.*

PROOF: By Theorem 2.8, $E'(f) = \gamma_1(f')E'(z) + \gamma_2(f'')E(z)$, $f \in C^n(I)$ where $E : C^n(I) \longrightarrow M$ is a continuous derivation such that

$$E'(fg) = \rho(f)E'(g) + \gamma_1(f')E(g) + \gamma_1(g')E(f) + \rho(g)E'(f).$$

Since $E'(z) \in W_1$, $p \longrightarrow \gamma_2(p'')E(z)$ is continuous in $C^n(I)$. By Theorem 1.1,

$$E(z) \in W_2.$$

Since $D'(z^2) = \rho(2z)D'(z) + 2D(z) = \rho(2z)E'(z) + 2E(z)$,

$$D(z) = E(z) \in W_2.$$

4. The continuity ideals and ranges of second order derivations

THEOREM 4.1. *Let $D' : C^n(I) \longrightarrow M$ be a continuous second order derivation with primitive D. If $D'(z), D(z) \in W_k$, for some k, $1 \leq k \leq n$, then*

$$D'(f) \in W_k, \quad f \in C^n(I).$$

PROOF: Since D' is continuous and $D'(z) \in W_k$, $k = 1, 2, \ldots, n$,

$$D'(f) = \gamma_1(f')D'(z) + \gamma_2(f'')D(z), \quad f \in C^n(I).$$

By Theorem 1.2,

$$\gamma_1(f')D'(z), \gamma_2(f'')D(z) \in W_k.$$

LEMMA 4.2. *Let $D' : C^n(I) \longrightarrow M$ be a second order derivation with primitive D. Then D' is separable.*

PROOF:

$$\begin{aligned}\|D'(fg)\| &= \|\rho(f)D'(g) + \gamma_1(f')D(g) + \gamma_1(g')D(f) + \rho(g)D'(f)\| \\ &\leq \|\rho\|\,\|f\|_n\|D'(g)\| + n\|\gamma_1\|\,\|f\|_n\|D(g)\| \\ &+ n\|\gamma_1\|\,\|g\|_n\|D(f)\| + \|\rho\|\,\|g\|_n\|D'(f)\|.\end{aligned}$$

Hence

$$\|D'(fg)\| \leq C(\|f\|_n + \|D(f)\| + \|D'(f)\|)(\|g\|_n + \|D(g)\| + \|D'(g)\|)$$

where $C = \max\{\|\rho\|, n\|\gamma_1\|\}$.

THEOREM 4.3. *Let $D' : C^n(I) \longrightarrow M$ be a discontinuous second order derivation with primitive D. Then there exists a finite set $F = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ in I such that*

$$\cap_{i=1}^m M_{n,n}(\lambda_i) \subset \Im(D').$$

PROOF: Suppose F =hull($\Im(D')$) is not finite. Then it is possible to select a sequence of points $\{\lambda_i\} \subset F$ such that no λ_i is a cluster point of F. Also, we may select open disjoint sets U_i and V_i such that $\lambda_i \in V_i \subset V_i^- \subset U_i$, $i = 1, 2, \ldots$ We can choose $f_i \in C^n(I)$ such that $f_i(\lambda_i) = 1$ on V_i and $f = 0$ on U_i^c. So $f_i^2 \notin \Im(D')$. We may select unit vectors $g_n \in C^n(I)$ such that

$$n\|f_n\|^2 \leq \|D'(f_n^2 g_n)\|.$$

Since $f_n f_m g_m = 0$ for $n \neq m$ this contradicts Theorem 1.4. Hence F is a finite set.

References

[1] W. G. Bade and P. C. Curtis, Jr., *The continuity of derivation of Banach algebra*, J. Funct. Anal. **16** (1974), 372–387.

[2] W. G. Bade and P. C. Curtis, Jr., *The structure of module derivations of Banach algebra of differentiable functions*, J. Funct. Anal. **28** (1978), 226–247.

[3] The semi-simplicity manifold of arbitrary operators, Trans. Amer. Math. Soc. **123** (1966), 241–252.

[4] S. Kantorovitz, "Spectral theory of Banach space operators," Springer-Verlag, Berline and New York, 1983.

[5] Dal-Won Park, *The image of derivations on Banach algebras of differential functions*, Chungcheong Math. Soc. **2** (1989), 81–90.

[6] K. B. Laursen and J. D. Stein Jr., *Automatic continuity in Banach spaces and algebras*, Amer. J. Math. **95** (1974), 485–506.

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