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Some Geometric Properties of the Weak*-integral

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ABSTRACT. We prove that if a weak*-measurable function f defined on a finite measure space into a dual Banach space is separable-like, then for every measurable set E, the weak* core of f over E is the weak* convex closed hull of the weak* essential range of f over E.

0. Introduction

Let (Ω, Σ, μ) be a finite measure space and X a Banach space with continuous dual X^*. An X^*-valued function f defined on Ω is said to be weak*-measurable if $\hat{x} \circ f$ is measurable for each $x \in X$. A weak*-measurable function f is said to be weak*-integrable if $\hat{x} \circ f$ is integrable for each $x \in X$, and the weak*-integral of f over $E \in \Sigma$ means the element x_E^* of X^* such that $x_E^*(x) = \int_E \hat{x} \circ f \, d\mu$ for all $x \in X$. We write $x_E^* = (w^*) - \int_E f \, d\mu$.

In the present paper, we consider weak* cores of weak*-measurable functions (cf. The definition of [1]) and define weak* essential ranges of weak*-measurable functions (cf. Definition 1.2. of [7]), and investigate their properties. For a subset A of X^*, the weak* closure of A in X^* and the weak* closed convex hull of A in X^* are denoted by $\overline{A}^{w^*}$ and $\overline{co}^{w^*}(A)$, respectively.

1. The weak* core

We consider a weak* analogy of the core in [4] and investigate its properties.

DEFINITION 1.1: Let $f : \Omega \to X^*$ be a weak*-measurable function and E a measurable set. The weak* core of f over E is defined, denoted cor $^*_f(E)$, to be the subset of X^* given by the formula

$$\text{cor }^*_f(E) = \bigcap_{\mu A = 0} \overline{co}^{w^*}(f(E \backslash A)).$$

We give two lemmas which provide the basis for the section.

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LEMMA 1.2. (The mean value theorem for the weak*-integral) *Let $f : \Omega \to X^*$ be weak*-integrable. Then for each measurable set E of positive measure*

$$\frac{1}{\mu(E)}\left((w^*) - \int_E f\, d\mu\right) \in \overline{co}^{w^*}(f(E)).$$

PROOF: Suppose there is a measurable set E of positive measure such that

$$\frac{1}{\mu(A)}\left((w^*) - \int_E f\, d\mu\right) \notin \overline{co}^{w^*}(f(E)),$$

then by the Hahn-Banach theorem [**8**, Theorem 3.4.(b)] and the fact that $(X^*, \text{weak}^*)^* = X$ [**2**, Theorem V.1.3.], there exist an x in X, a real number α such that

$$\frac{1}{\mu(A)}\int_E \hat{x} \circ f\, d\mu < \alpha < f(w)x$$

for all w in E.

Integrating over E,

$$\int_E \hat{x} \circ f\, d\mu < \alpha\mu(E) < \int_E \hat{x} \circ f\, d\mu,$$

a contradiction. Q.E.D.

LEMMA 1.3. *Let X be a locally convex topological vector space and A a subset of X. Then x is a member of $\overline{co}(A)$ if and only if for every $x^* \in X^*$, $x^*(x) \geq \inf\{x^*(a) : a \in A\}$.*

Now we prove in details (i), (ii) and (iii) of proposition 1 of [**1**].

THEOREM 1.4. *Let $f : \Omega \to X^*$ be a weak*-integrable function. Then*

(a) *for every measurable set E of positive measure,* cor $^*_f(E)$ *is not empty.*
(b) *for every measurable set E of positive measure,* cor $^*_f(E) = \overline{co}^{w^*}\left\{\frac{1}{\mu(B)}\left((w^*) - \int_B f\, d\mu\right) : B \subset E, \mu(B) > 0\right\}$.

PROOF: The proof of (a) is omitted. For the proof of (b), let B be a subset of positive measure of E, A a null set. Applying Lemma 1.2, $\frac{1}{\mu(B)}\left((w^*) - \int_B f\, d\mu\right) \in \overline{co}^{w^*}(f(B\backslash A))$, which is contained in $\overline{co}^{w^*}(f(E\backslash A))$. Hence cor $^*_f(E)$ contains

$$\overline{co}^{w^*}\left\{\frac{1}{\mu(B)}\left((w^*) - \int_B f\, d\mu\right) : B \subset E, \mu(B) > 0\right\}.$$

For the opposite direction, let x^* be an element of cor $^*_f(E)$, and x in X. For any $\varepsilon > 0$, choose a countable partition π of E and a function ϕ, constant on the set in π, such that the inequality $|f(w)x - \phi(w)| < \frac{\varepsilon}{4}$ holds for all w in E. Note that if B is any set in π with positive measure and w in B then we have the inequality.

$$\left|f(w)x - \frac{1}{\mu(B)}\int_B \hat{x} \circ f\, d\mu\right| < \frac{\varepsilon}{2}.$$

Let A be the union of the null sets in π. Since $x^* \in \overline{co}^{w^*}(f(E\backslash A))$, there exists a finite convex sum $\sum t_i f(w_i)$ with $w_i \in E\backslash A$ such that

$$|x^*(x) - \sum t_i f(w_i)x| < \frac{\varepsilon}{2}.$$

Finally, for each number i let B_i be the set in π containing w_i. Observe that

$$\left|x^*(x) - \sum t_i \frac{1}{\mu(B_i)}\int_{B_i} \hat{x} \circ f\, d\mu\right| < \varepsilon.$$

Since ε is arbitrary, we have that

$$x^*(x) \geq \inf\left\{\frac{1}{\mu(B)}\int_B \hat{x} \circ f\, d\mu : B \subset E, \mu(B) > 0\right\}.$$

It follows from Lemma 1.3 that x^* is an element of

$$\overline{co}^{w^*}\left\{\frac{1}{\mu(B)}\left((w^*) - \int_B f\, d\mu\right) : B \subset E, \mu(B) > 0\right\}. \text{Q.E.D.}$$

DEFINITION 1.5: Let f, $g : \Omega \to X^*$ be weak*-measurable. f and g are said to be weak* equivalent if for each x in X, $\hat{x} \circ f = \hat{x} \circ g$ almost everywhere.

Using the notion of the essential supremum, one can immediately verify following lemma.

LEMMA 1.6. *Let α and β be real-valued and measurable and E a measurable set of positive measure. If $\alpha(w) < \beta(w)$ for all w in E, then there exists a set $E' \subset E$ such that $\mu(E') > 0$ and*

$$\sup \alpha(E') < \inf \beta(E').$$

The proof of the lemma is omitted.

THEOREM 1.7. *Let $f, g : \Omega \to X^*$ be weak*-measurable functions and E a measurable set. Then for each x in X, $\hat{x} \circ f = \hat{x} \circ g$ a.e. if and only if* cor $^*_f(E) =$ cor $^*_g(E)$ *for every measurable set E.*

PROOF: Suppose that $\hat{x} \circ f = \hat{x} \circ g$ a.e. for all x in X. Let X^* be any element in cor $^*_f(E)$, and let A_0 be a null set. We will show that for each x in X,

$$x^*(x) \geq \inf\{g(w)x : w \in E \backslash A_0\}.$$

To this end, fix x and let $A_1 = A_0 \cup \{w : f(w)x \neq g(w)x\}$. Clearly $\mu(A_1) = 0$, hence x^* is an element of $\overline{co}^{w^*}(f(E\backslash A_1))$ and

$$\begin{aligned} x^*(x) &\geq \inf\{f(w)x : w \in E\backslash A_1\} \\ &= \inf\{g(w)x : w \in E\backslash A_1\} \\ &\geq \inf\{g(w)x : w \in E\backslash A_0\}. \end{aligned}$$

For the converse, suppose there is an x in X such that the condition $\hat{x} \circ f = \hat{x} \circ g$ a.e. fails. We may assume without loss of generality that

$$\mu\{w : f(w)x < g(w)x\} > 0.$$

By Lemma 1.6, there must be a set E' of positive measure such that $\sup\{f(w)x : w \in E'\} < \inf\{g(w)x : w \in E'\}$.

The sets $\overline{co}^{w^*}(f(E'))$ and $\overline{co}^{w^*}(g(E'))$ are thus disjoint, and the equality cor $^*_f(E) =$ cor $^*_g(E)$ can only hold the weak* cores of f and g are empty. Q.E.D.

2. The weak* essential range

DEFINITION 2.1: Let f be a weak*-measurable function from Ω into X^* and $E \in \sum$. We define the weak* essential range of f over E, denoted $r_f^*(E)$, to be the set of those $x^* \in X^*$ such that for every $\varepsilon > 0$ and for every finite subset Δ of the ball of X (denoted by Ball(X)), the measure of $\bigcap_{x\in\Delta}\{w \in E : |\hat{x} \circ f(t) - x^*(x)| < \varepsilon\}$ is strictly positive.

REMARK: One can immediately show that $x^* \in r_f^*(E)$ if and only if for every $\varepsilon > 0$ and for every finite subset Δ of X,

$$\mu\Big(\bigcap_{x\in\Delta}\{w \in E : |\hat{x} \circ f(t) - x^*(x)| < \varepsilon\}\Big) > 0.$$

And it is clear that if f and g are weak* equivalent then $r_f^*(E) = r_g^*(E)$ for each $E \in \sum$.

PROPOSITION 2.2. *If $f : \Omega \to X^*$ is weak*-measurable and $E \in \sum$, then*

(a) *$r_f^*(E)$ is a weak* closed subset of X^*.*
(b) *If $\mu(E) = 0$, then $r_f^*(E) = \emptyset$.*
(c) *$E \subset F$ implies $r_f^*(E) \subset r_f^*(F)$, E, $F \in \sum$.*

PROOF: (a) Let $x^* \notin r_f^*(E)$. Then there exist an $\varepsilon > 0$ and a finite $\Delta \subset$ Ball(X), $\mu\left(\bigcap_{x\in\Delta}\{w \in E : |x_0^*(x) - \hat{x} \circ f(w)| < \varepsilon\}\right) = 0$. Let $x_1^* \in \bigcap_{x\in\Delta}\{x^* \in X^* : |x_0^*(x) - x^*(x)| < \frac{\varepsilon}{2}\}$. Then

$$\begin{aligned}
&\mu\Big(\bigcap_{x\in\Delta}\{w \in E : |x_1^*(x) - \hat{x} \circ f(w)| < \frac{\varepsilon}{2}\}\Big)\\
=&\mu\Big(\bigcap_{x\in\Delta}\{w \in E : |x_1^*(x) - \hat{x} \circ f(w)| + \frac{\varepsilon}{2} < \varepsilon\}\Big)\\
\leq&\mu\Big(\bigcap_{x\in\Delta}\{w \in E : |x_1^*(x) - \hat{x} \circ f(w)| < \varepsilon\}\Big) = 0.
\end{aligned}$$

Therefore $x_1^* \notin r_f^*(E)$ and $r_f^*(E)$ is weak* closed.

(b), (c) are clear from the definition. Q.E.D.

THEOREM 2.3. *If $\mu(E) > 0$, $f : \Omega \to X^*$ is norm-separable valued weak*-measurable, then $r_f^*(E) \cap f(E)$ is not empty.*

PROOF: Suppose $r_f^*(E) \cap f(E) = \emptyset$, then for every $v \in E$, there exist a finite $\Delta_v \subset \mathrm{Ball}(X)$ and an $\varepsilon_v > 0$ such that

$$\mu\Big(\bigcap_{x \in \Delta_v} \{w \in E : |\hat{x} \circ f(w) - \hat{x} \circ f(v)| < \varepsilon_v\}\Big) = 0.$$

Since $f(E)$ is norm separable, there exists a sequence $\langle v_n \rangle$ in E such that $f(E) \subset \bigcup_{n=1}^{\infty} B(f(v_n), \varepsilon_{v_n})$. Therefore

$$E \subset \bigcup_{n=1}^{\infty}\Big(\bigcap_{x \in \Delta_{v_n}} \{w \in E : |\hat{x} \circ f(w) - \hat{x} \circ f(v_n)| < \varepsilon_{v_n}\}\Big),$$

and $\mu(E) = 0$, which contradicts to our assumption. Q.E.D.

COROLLARY 2.4. *Under same hypothesis in Theorem 2.3, $N = \{w \in E : f(w) \notin r_f^*(E)\}$ is a null set.*

PROOF: If $\mu(N) > 0$, there exists an $w \in E$ such that $f(w) \in r_f^*(N) \subset r_f^*(E)$ which contradicts the definition of N. Q.E.D.

THEOREM 2.5. *Let $f : \Omega \to X^*$ be norm separable and weak*-measurable and let $E \in \sum$. Then $r_f^*(E) = \bigcap_{\mu A = 0} \overline{f(E \backslash A)}^{w^*}$.*

PROOF: If $\mu(E) = 0$, it is clear from Definition 2.1. Fix $E \in \sum$ with $\mu(E) > 0$. Let $x_0^* \in r_f^*(E)$ and A any null set. For given $\varepsilon > 0$ and a finite $\Delta > X$, we know that the set

$$\bigcap_{x \in \Delta} \{w \in E : |\hat{x} \circ f(w) - x_0^*(x)| < \varepsilon\}$$

has positive measure (see Remark).

In particular, there exists $w_{(\varepsilon,\Delta)} \in E \backslash A$ such that

$$|\hat{x} \circ f(w_{(\varepsilon,\Delta)}) - x_0^*(x) < \varepsilon \text{ for all } x \in \Delta.$$

Hence $x_0^* \in \overline{f(E \backslash A)}^{w^*}$. Since this holds for each mull set A, we get $x^* \in \bigcap_{\mu A = 0} \overline{f(E \backslash A)}^{w^*}$.

Conversely suppose $\mu(E) > 0$ and $x_0^* \in \bigcap_{\mu A=0} \overline{f(E\backslash A)}^{w^*}$. For any given $\Delta \subset \text{Ball}(X)$ and $\varepsilon > 0$, let $A_{(\varepsilon,\Delta)} = \bigcap_{x\in\Delta}\{w \in E : |\hat{x}\circ f(w) - x_0^*(x)| < \varepsilon\}$. Then clearly $x_0^* \notin \overline{f(E\backslash A_{(\varepsilon,\Delta)})}^{w^*}$. Therefore $\mu(A_{(\varepsilon,\Delta)}) > 0$. Since this holds for each finite $\Delta \subset \text{Ball}(X)$ and for each $\varepsilon > 0$, $x_0^* \in r_f^*(E)$. Q.E.D.

COROLLARY 2.6. *Under same hypothesis Theorem 2.5, if* $E \in \sum$, *then* $r_f^*(E) = \overline{f(E\backslash N)}^{w^*}$ *where* $N = \{w \in E : f(w) \notin r_f^*(E)\}$.

PROOF: Since $f(E\backslash N) \subset r_f^*(E)$ and r_f^* is weak* closed, $\overline{f(E\backslash N)}^{w^*} \subset r_f^*(E)$. The opposite direction is trivial. Q.E.D.

THEOREM 2.7. *Let* $f : \Omega \to X^*$ *be norm separable and weak*-measurable. Then* cor $_f^*(E) = \overline{co}^{w^*}(r_f^*(E))$ *for all* $E \in \sum$.

PROOF: Since $\overline{f(E\backslash A)}^{w^*} \subset \overline{co}^{w^*}(f(E\backslash A))$ for each null set A, and cor $_f^*(E)$ is weak* closed convex, we get $\overline{co}^{w^*}(r_f^*(E)) \subset$ cor $_f^*(E)$.

Conversely, since there is a null set A_0 such that $r_f^*(E) = \overline{f(E\backslash A_0)}^{w^*}$, cor $_f^*(E) = \bigcap_{\mu A=0} \overline{co}^{w^*}(f(E\backslash A_0)) \subset \overline{co}^{w^*}(f(E\backslash A_0)) = \overline{co}^{w^*}(r_f^*(E))$. Q.E.D.

For our main result, we define a terminology as an analogy of Huff[**5**].

DEFINITION 2.8: A weak*-measurable function $f : \Omega \to X^*$ is separable-like provide there exists a norm separable subspace D of X^* such that for every $x \in X$, $\hat{x}\circ(\chi_D f) = \hat{x}\circ f$ a.e. where χ_D is the characteristic function of D.

THEOREM 2.9. *If a weak*-measurable function* $f : \Omega \to X^*$ *is separable-like, then* cor $_f^*(E) = \overline{co}^{w^*}(r_f^*(E))$ *for all* $E \in \sum$.

PROOF: The result follows immediately from Theorem 1.7, Theorem 2.7, and Remark under Definition 2.1. Q.E.D.

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