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On the Asymptotic Martingales of Weakly Measurable Pettis Integrable Random Variables

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ABSTRACT. In this paper, we obtain conditions which are sufficient to insure convergence of these amarts in the weak topology of E.

1. Introduction

The concept of an asymptotic martingale, abbreviated as amart, was first given by Meyer [10] who proved that a continuous parametered scalar-valued amart converges almost everywhere if it is essentially bounded. Austin, Edgar, and Tulcea [1] proved that "a real-valued amart converges almost everywhere if it is L_1-bounded". Chatterji [5] proved that if a Banach space E has the Randon-Nikodym property, then every E-valued L_1-bounded martingale converges almost everywhere. Chacon and Sucheston [3] proved that if E is a Banach space with a separable dual and has the Randon-Nikodym property then an E-valued amart (f_n) converges almost everywhere in the weak topology of E if the condition $\sup_{\tau\in T}\int|f_\tau|\,dp<\infty$ holds.
In this paper, we obtain conditions which are sufficient to insure convergence of these amarts in the weak topology of E.

2. Elementary notations

A sequence of integrable functions (f_n) defined on a probability space $(S,\sum,P)$ and adapted to an increasing sequence of sub-σ-fields $(\sum_n)$ is called an asymptotic martingale, abbreviated as amart, if $\int|f_n|\,dp<\infty$ for all $n\in N$ and $\lim_{\tau\in T}\int f_\tau\,dp$ exists, where N is the set of all natural numbers, T is the collection of all bounded stopping times for $(\sum_n)$ and the limit on T is with respect to the usual ordering on T.

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A martingale is a special case of amart. A function $f : S \to R$ is $\sum$-measurable if for every real value a, the set $\{s \in S : f(s) < a\}$ is in $\sum$.

On a probability space, a $\sum$-measurable function will be called a random variable. A random variable is also a measurable function on that probability space.

Let (f_n) be a sequence of random variables and $(\sum_n)$ an increasing sequence of sub-σ-fields of $\sum$. For example, $\sum_n$ could be the σ-field generated by $\{f_1, f_2, \ldots, f_n\}$ for $n \in \mathrm{N}$. The sequence (f_n) is said to be adapted to $(\sum_n)$ if f_n is $\sum_n$-measurable for each $n \in \mathrm{N}$. A martingale $(f_n, \sum_n, n \in \mathrm{N})$ is a sequence (f_n) of integrable random variables adapted to an increasing sequence $(\sum_n)$ of sub-σ-fields of $\sum$ such that

$$E\{f_n | \textstyle\sum_m\} = f_m$$

for every $m, n \in \mathrm{N}$ with $m \leq n$. A stopping time is a random variable $\tau : S \to \{0, 1, \ldots, \infty\}$ such that for each $n \in \mathrm{N}$, $\{\tau = n\} \in \sum_n$.

A function f from S to E is a weakly measurable random variable if for each $x^* \in E^*$ the complex valued function $x^*f(s) = x^*(f(s))$ is a measurable function. A random variable $f : S \to E$ is Pettis integrable, sometimes called weakly integrable, if for $x^* \in E^*$, $\int |x^*f| \, dp < \infty$ and for each $A \in \sum$ there is an element of E, denoted by $\int_A f \, dp$, such that

$$x^* \left(\int_A f \, dp \right) = \int_A x^* f \, dp$$

for every $x^* \in E^*$. If a sequence (f_n) of weakly measurable and Pettis integrable random variables is adapted to $(\sum_n, n \in \mathrm{N})$ then $(f_n, \sum_n : n \in \mathrm{N})$ will be called an amart if $\lim_{\tau \in T} \int f_\tau \, dp$ exists with respect to the topology of E.

3. Main theorems

PROPOSITION 1. *If $(f_n, \sum_n : n \in \mathrm{N})$ is a real-valued amart such that*

$$\sup_{n \in \mathrm{N}} \int |f_n| \, dp < \infty,$$

then the sequence (f_n) *converges almost everywhere to an integrable random variable* f.

PROOF: Since $\left(\int f_\tau\right)_{\tau \in T}$ converges, $\left(\int f_\tau^+\right)_{\tau \in T}$ and $\left(\int f_\tau^-\right)_{\tau \in T}$ converge. Hence we may assume, without loss of generality, $f_n \geq 0$ for all $n \in \mathbf{N}$. Suppose that (f_n) does not converge almost everywhere. Then there are real numbers $\alpha < \beta$ such that $P(A) > 0$, where

$$A = \{s \mid \liminf f_n(s) < \alpha < \beta < \limsup f_n(s)\}.$$

We will show that $\left(\int f_\tau\right)_{\tau \in T}$ is not Cauchy. For given

$$\varepsilon = \frac{(\beta - \alpha)P(A)}{2}$$

we will show that given any integer $M \geq 1$ there exist bounded stopping times $\tau_1 \geq M$, $\tau_2 \geq M$ with

$$\int f_{\tau_2} - \int f_{\tau_1} \geq \varepsilon.$$

Let $\delta = \frac{\varepsilon}{2\beta}$ and let $M \geq 1$ be any integer. There exists a set B and an integer $N \geq M$ such that $B \in F_N$ and $P(A \Delta B) \leq \delta$. There exist integers $N'' > N' > N$ such that if

$$\Omega_0 = \{s \mid \inf_{N \leq n \leq N'} f_n(s) < \alpha < \beta < \sup_{N' \leq n \leq N''} f_n(s)\}$$

then $P(A - \Omega_0) \leq \delta$. Define

$$C_1 = \{s \in B \mid \inf_{N \leq n \leq N'} f_n(s) < \alpha\}$$
$$C_2 = \{s \in C_1 \mid \sup_{N' \leq n \leq N''} f_n(s) > \beta\}.$$

then we have $C_1 \in F_{N'}$, $C_2 \in F_{N''}$, $C_2 \subset C_1 \subset B$,

(1) $$P(C_2) \geq P(A) - 2\delta, \quad \text{and} \quad P(C_1 - C_2) \leq 2\delta.$$

Define stopping time τ_1, τ_2 by

$$\tau_1(s) = \begin{cases} N' & s \notin C_1 \\ \inf\{n \mid N \le n \le N', f_n(s) < \alpha\} & s \in C_1, \end{cases} \quad \text{and}$$

$$\tau_2(s) = \begin{cases} N' & s \notin C_1 \\ N'' & s \in C_1 - C_2 \\ \inf\{n \mid N' \le n \le N'', f_n(s) < \beta\} & s \in C_2. \end{cases}$$

Then $M \le N \le \tau_1 \le \tau_2$ and by (1) we have

$$\begin{aligned} \int f_{\tau_2} - \int f_{\tau_1} &= \int (f_{\tau_2} - f_{\tau_1}) = \int_{c_1} (f_{\tau_2} - f_{\tau_1}) \\ &= \int_{c_2} (f_{\tau_2} - f_{\tau_1}) + \int_{c_1 - c_2} f_{\tau_2} - \int_{c_1 - c_2} f_{\tau_1} \\ &\ge (\beta - \alpha)P(c_2) + o - \alpha p(c_1 - c_2) \\ &= (\beta - \alpha)(P(A) - 2\delta) - 2\delta\alpha \\ &= (\beta - \alpha)P(A) - 2\delta\beta = \varepsilon. \end{aligned}$$

THEOREM 2. *If the amart $(f_n, \sum_n : n \in \mathbf{N})$ is uniformly integrable then the sequence (f_n) converges to an integrable random variable f in the L_1-norm as well as almost everywhere.*

PROOF: Since the amart $(f_n, \sum_n : n \in \mathbf{N})$ is uniformly integrable, that is, if for any $\varepsilon > 0$, there is a real number b such that

$$\int_{\{f_n > b\}} |f_n| \, dp < \varepsilon,$$

for all $n \in \mathbf{N}$, then the Vitali convergence theorem implies that in addition to converging almost everywhere the sequence (f_n) converges in the L_1-norm to an integrable random variable f.

COROLLARY 3. *Let Λ be a directed set and $(f_\alpha, \sum_\alpha : \alpha \in \Lambda)$ a real-valued amart with $\sup_{\alpha \in \Lambda} \int |f_\alpha| \, dp < \infty$. Then the net $(f_\alpha : \alpha \in \Lambda)$ converges in probability.*

THEOREM 4. *Assume that E is a locally convex linear topological space with a separable dual. Let $(f_n, \sum_n : n \in \mathbf{N})$ be a weakly measurable, weakly integrable amart from S to E such that the following conditions are true:*

(a) *for each $x^* \in E^*$, $\sup_{n \in \mathbf{N}} \int |x^* f_n| \, dp < \infty$,*

(b) *there exists the closed separable linear manifold* E_0 *spanned by the value of* $f_n(s)$, $s \in S$, $n \geq 1$.

Then there is a weakly measurable, Pettis integrable random variable f_∞ *such that for almost all* $s \in S$,

$$\lim_n x^* f_n(s) = x^* f_\infty(s)$$

for every $x^* \in E^*$.

PROOF: For each $x^* \in E^*$, $(x^* f_n, \sum_n : n \in \mathbf{N})$ is an amart such that $\sup_{n \in \mathbf{N}} \int |x^* f_n| \, dp < \infty$. Therefore by Proposition 1 the sequence $(x^* f_n)$ converges almost everywhere to an integrable random variable h^*_x.

We will show the existence of a random variable $f_\infty : S \to E$ such that for all $x^* \in E^*$, and almost all $s \in S$, $x^* f_\infty(s) = h_{x^*}(s)$. By condition (b), E_0 is weakly closed, by a general theorem [**6**, p.422, theorem 13], the subset of E_0 consisting of the value of $f_n(s)$ is relatively weakly compact and so there exists a subsequence (n_k) can be chosen so that $f_{n_k}(s)$ converges weakly to $f_\infty(s)$, an element of E_0. Now for any $x^* \in E^*$ the sequence $x^* f_n$ converges a.e.. Hence $\lim_{n \to \infty} x^* f_n(s) = x^* f_\infty(s) a.e..$ We have $h_{x^*}(s) = \lim_n x^* f_n(s) = x^* f_\infty(s) a.e..$

Let $A = \{x^*_k : k \in \mathbf{N}\}$ be a dense subset of E^* in the weak topology of E^*. Then for each $x^*_k \in A$ there is a P-null set F_k such that if $s \notin F_k$ then

$$h_{x^*_k}(s) = \lim_n (x^*_k f_n(s)) = x^*_k f_\infty(s).$$

Thus if s is not in the null set $F = \bigcup_{k \in \mathbf{N}} F_k$, then

$$h_{x^*_k}(s) = x^*_k f_\infty(s)$$

for $x^*_k \in A$.

Since A is dense in the week-topology of E^*, for any $\varepsilon > 0$ and $x^* \in E^*$ if $s \notin F$, there is an $x^*_k \in A$ such that x^*_k is contained in the neighborhood

$$\{y^* : |y^*(t) - x^*(t)| < \varepsilon/3 \text{ for all } t \in E_0\}$$

of x^*. Choose an integer N such that if $n \geq N$ then

$$|x_k^* f_n(s) - x_k^* f(s)| < \varepsilon/3.$$

Then, for $n \geq N$, we get

$$\begin{aligned} |x^* f_n(s) - x^* f_\infty(s)| \leq & |x^* f_n(s) - x_k^* f_n(s)| \\ & + |x_k^* f_n(s) - x_k^* f_\infty(s)| \\ & + |x_k^* f_\infty(s) - x^* f_\infty(s)| \\ & < \varepsilon. \end{aligned}$$

Thus for all $x^* \in E^*$ and almost all $s \in S$ we have

$$h_{x^*}(s) = \lim_n x^* f_n(s) = x^* f_\infty(s),$$

and so the proof is complete.

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