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A Decomposition of Positive Linear Operators on the Ordered Space of 2 × 2 Hermitian Matrices into a Sum of Four Extreme Operators

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1. Introduction

We denote E to be the real ordered space of all 2×2 Hermitian matrices with the positive cone consisting of all elements having nonnegative eigenvalues. A linear operator T on E is said to be positive if $T(P) \geq 0$ for every $P \geq 0$, and T is extreme if $S = \lambda T$ for some $\lambda \geq 0$ whenever $0 \leq S \leq T$.

It is proved in [1] that a positive linear operator T on E is extreme if and only if it is unitarily equivalent to a map of the form $S_{\vec{z}}$ for some $\vec{z} \in \mathbb{C}^2$. The linear operator $S_{\vec{z}}$ is defined by $S_{\vec{z}}(\vec{x}\vec{x}^*) = \vec{w}\vec{w}^*$ for every $\vec{x} \in \mathbb{C}^2$ where $w_i = x_i z_i$, $i = 1, 2$.

We know by the Krein-Milman theorem that every positive linear operator on E is a convex combination of extreme operators. But in this paper, we prove that every positive linear operator on E can be decomposed into a sum of four extreme operators. Note that the dimension of E is four and hence the vector space of all linear operators on E has dimension sixteen.

In the following, we denote E_{ii} for $\vec{e}_i\vec{e}_i^T$, E_{12} for $\vec{e}_1\vec{e}_2^T + \vec{e}_2\vec{e}_1^T$ and $\widehat{E}_{12}$ for $i\vec{e}_1\vec{e}_2^T - i\vec{e}_2\vec{e}_1^T$ where $\vec{e}_1 = (1,0)^T$, $\vec{e}_2^T = (0,1)$. The unit matrix $E_{11} + E_{22}$ will be denoted by I while I will also be used for the identity operator on E. Recall that every element of E can be written as $\lambda\vec{x}\vec{x}^* + \mu\vec{y}\vec{y}^*$ for some λ, $\mu \in \mathbb{R}$ and an orthonormal set $\{\vec{x}, \vec{y}\}$ of eigenvectors. A linear operator T is determined if $T(\vec{x}\vec{x}^*)$ is defined for every $\vec{x} \in \mathbb{C}^2$ and hence if $T\begin{pmatrix} 1 & re^{i\theta} \\ re^{-i\theta} & r^2 \end{pmatrix}$ is defined for all $r \geq 0$, $\theta \in \mathbb{R}$ along with $T(E_{22})$.

If Q is an arbitrary nonsingular matrix, then we define a linear operator by $S_Q(A) = QAQ^*$ for all $A \in E$. Note that $S_Q^{-1} = S_{Q^{-1}}$.

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When U is a unitary matrix, we write simply U instead of S_U. In case $U = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$ for $\alpha \in \mathbb{R}$, then we have

$$S_U \begin{pmatrix} a & b+ci \\ b-ci & d \end{pmatrix} = U \begin{pmatrix} a & b+ci \\ b-ci & d \end{pmatrix} U^* = \begin{pmatrix} a & (b+ci)e^{i\alpha} \\ (b-ci)e^{-i\alpha} & d \end{pmatrix}$$

for all a, b, c, $d \in \mathbb{R}$. We write this operator as S_α instead of S_U or U. Note that we have $S_\alpha^{-1} = S_{-\alpha}$.

THEOREM 1.1. *Let T be a positive linear operator on E. Then T is extreme if and only if there exist unitary matrices U, V and $\vec{z} \in \mathbb{C}^2$ such that $T = U \circ S_z \circ V$ or $\overline{T} = U \circ S_z \circ V$.*

COROLLARY 1.2. *Let T be a nonzero positive linear operator on E. Then T is extreme if and only if T maps every extreme point of E to either 0 or another extreme point.*

The proof of Theorem 1.1 and Corollary 1.2 above are given in [**1**; 5.1, 5.2]. We quoted them here as they are used in the following sections.

COROLLARY 1.3. *Let Q be an arbitrary 2×2 matrix, W be a unitary matrix, $\vec{z}$, $\vec{w} \in \mathbb{C}^2$, and $T = S_{\vec{z}} \circ W \circ S_{\vec{w}}$. Then*

(a) $S_Q = U_1 \circ S_{\vec{x}} \circ V_1$ *or* $\overline{S}_Q = U_1 \circ S_{\vec{x}} \circ V_1$

(b) $T = U_2 \circ S_{\vec{y}} \circ V_2$ *or* $\overline{T} = U_2 \circ S_{\vec{y}} \circ V_2$

for some unitary matrices U_i, V_i, $i = 1, 2$, and $\vec{x}$, $\vec{y} \in \mathbb{C}^2$.

PROOF: Note that both S_Q and T map extreme points to zero or other extreme points. Therefore, they are extreme operators by Corollary 1.2 and hence Theorem 1.1 applies.

2. Preliminaries

In this section, we consider some of the elementary results that are necessary for the proofs in section 3. We also prove that a positive linear operator T on E with $\dim(\operatorname{Ker} T) \geq 2$ is a sum of four extreme operators.

We quote the following two Lemmas from **[1]** without proof as they are used frequently in what follows.

LEMMA 2.1. *If $\{\vec{x}, \vec{y}\}$ is a linearly independent set in $\mathbb{C}^2$, then there exists a nonsingular matrix Q such that $S_Q(\vec{x}\vec{x}^*) = E_{11}$, $S_Q(\vec{y}\vec{y}^*) = E_{22}$.*

LEMMA 2.2. *Let T be a positive linear operator on E with $\dim(\operatorname{Ker} T) = 2$. Then there exist unitary matrices U and V such that for $S = U \circ T \circ V$, we have $(\operatorname{Ker} S)^\circ = \operatorname{Span}\{\vec{x}\vec{x}^*, \vec{y}\vec{y}^*\}$, $S(E) = \operatorname{Span}\{\vec{z}\vec{z}^*, \vec{w}\vec{w}^*\}$ for some $\vec{x}$, $\vec{y}$, $\vec{z}$, $\vec{w} \in \mathbb{C}^2$.*

LEMMA 2.3. *Let T be a positive linear operator on E with $T(\vec{x}\vec{x}^*) = \lambda_x \vec{\xi}\vec{\xi}^*$ for all $\vec{x} \in \mathbb{C}^2$ where $\vec{\xi}$ is fixed. then there exist unitary operators U, V_1, V_2 and $\vec{z}$, $\vec{w} \in \mathbb{C}^2$ such that $T = V_1 \circ S_{\vec{z}} \circ W + V_2 \circ S_{\vec{w}} \circ U$.*

PROOF: We define a linear functional on E by $f(\vec{x}\vec{x}^*) = \lambda_x$ where λ_x is from $T(\vec{x}\vec{x}^*) = \lambda_x \vec{\xi}\vec{\xi}^*$. Then f is clearly positive and hence there exists $0 \leq P \in E$ such that $f(A) = \langle P, A\rangle$ for all $A \in E$. We write $P = \alpha\vec{\eta}\vec{\eta}^* + \beta\vec{\zeta}\vec{\zeta}^*$ where $\{\vec{\eta}, \vec{\zeta}\}$ is orthonormal and let $U = (\vec{\eta}, \vec{\zeta})^*$, then $UPU^* = \alpha E_{11} + \beta E_{22}$. Now, we have $T(\vec{x}\vec{x}^*) = \langle P, \vec{x}\vec{x}^*\rangle \vec{\xi}\vec{\xi}^* = (\vec{x}^* P \vec{x})\vec{\xi}\vec{\xi}^* = \vec{x}^* U^*(\alpha E_{11} + \beta E_{22})U\vec{x}\vec{\xi}\vec{\xi}^* = \vec{y}^*(\alpha E_{11} + \beta E_{22})\vec{y}\vec{\xi}\vec{\xi}^* = (\alpha|y_1|^2 + \beta|y_2|^2)\vec{\xi}\vec{\xi}^*$, where $\vec{y} = U\vec{x}$. Let $S = T \circ U^*$.

Note that we may assume $\vec{\xi}^*\vec{\xi} = 1$ and we can take $\vec{\eta}' \in \mathbb{C}^2$ such that $\{\vec{\xi}, \vec{\eta}'\}$ is orthonormal. Let $V = (\vec{\xi}, \vec{\eta}')^*$, then $V \circ S(\vec{x}\vec{x}^*) = (\alpha|x_1|^2 + \beta|x_2|^2)E_{11} = \alpha S_{\vec{z}}(\vec{x}\vec{x}^*) + \beta W \circ S_{\vec{w}}(\vec{x}\vec{x}^*)$ where $\vec{z} = \vec{e}_1$, $\vec{w} = \vec{e}_2$, $W = E_{12}$. Therefore, we have $T = V^* \circ S_{\vec{z}} \circ U + V^* \circ W \circ S_{\vec{w}} \circ U$.

THEOREM 2.4. *Let T be a positive linear operator on E with $\dim(\operatorname{Ker} T) = 3$. Then T is a sum of four extreme operators.*

PROOF: Let $T(\vec{x}\vec{x}^*) = \lambda_x P$ where $0 \leq P \in E$. We write $P = \vec{\xi}\vec{\xi}^* + \vec{\eta}\vec{\eta}^*$ and define $T_1(\vec{x}\vec{x}^*) = \lambda_x \vec{\xi}\vec{\xi}^*$, $T_2(\vec{x}\vec{x}^*) = \lambda_x \vec{\eta}\vec{\eta}^*$ for all $\vec{x} \in \mathbb{C}^2$. Then we clearly have $0 \leq T_1, T_2$. By Lemma 2.3 above, both T_1 and T_2 are sums of 2 extreme operators and hence T is a sum of four extreme operators.

LEMMA 2.5. *Let S be a positive linear operator on E with $S(E) = \operatorname{Span}\{E_{11}, E_{22}\}$ and $S(E_{12}) = S(\widetilde{E}_{12}) = 0$. Then S is a sum of four extreme operators.*

PROOF: Let $S(E_{11}) = \alpha_1 E_{11} + \alpha_2 E_{22}$, $S(E_{22}) = \alpha_3 E_{11} + \alpha_4 E_{22}$ where $\alpha_i \geq 0$, $i = 1, 2, 3, 4$. Then for every $\vec{x} \in \mathbb{C}^2$, we have

$$\begin{aligned} S(\vec{x}\vec{x}^*) &= S(|x_1|^2 E_{11} + |x_2|^2 E_{22}) = |x_1|^2 S(E_{11}) + |x_2|^2 S(E_{22}) \\ &= |x_1|^2(\alpha_1 E_{11} + \alpha_2 E_{22}) + |x_2|^2(\alpha_3 E_{11} + \alpha_4 E_{22}) \\ &= (\alpha_1 S_{\vec{z}} + \alpha_2 V \circ S_{\vec{z}} + \alpha_3 V \circ S_{\vec{w}} + \alpha_4 S_{\vec{w}})(\vec{x}\vec{x}^*) \end{aligned}$$

with $\vec{z} = \vec{e}_1$, $\vec{w} = \vec{e}_2$, and $V = E_{12}$. Therefore, we have $S = \alpha_1 S_{\vec{z}} + \alpha_2 V \circ S_{\vec{z}} + \alpha_3 V \circ S_{\vec{w}} + S_{\vec{w}}$.

THEOREM 2.6. *Let T be a positive linear operator on E with $\dim(\operatorname{Ker} T) = 2$. Then T is a sum of four extreme positive operators.*

PROOF: By Lemma 2.2, there exist unitary operators U and V such that for $T_1 = U \circ T \circ V$, we have $(\operatorname{Ker} T_1)^\circ = \operatorname{Span}\{\vec{x}\vec{x}^*, \vec{y}\vec{y}^*\}$ and $T_1(E) = \operatorname{Span}\{\vec{z}\vec{z}^*, \vec{w}\vec{w}^*\}$. Now by Lemma 2.1, there exist S_Q, S_R such that $S_Q(E_{11}) = \vec{x}\vec{x}^*$, $S_Q(E_{22}) = \vec{y}\vec{y}^*$, $S_R(\vec{z}\vec{z}^*) = E_{11}$, $S_R(\vec{w}\vec{w}^*) = E_{22}$. Let $S = S_R \circ T_1 \circ S_Q$ then $S(E) = \operatorname{Span}\{E_{11}, E_{22}\}$ and $(\operatorname{Ker} S)^\circ = \operatorname{Span}\{E_{11}, E_{22}\}$. Therefore, by Lemma 2.5, S is a sum of four extreme operators and hence so is T.

3. Decomposition of Positive Linear Operators

In the previous section, we proved that any positive linear operator on E with $\dim(\operatorname{Ker} T) \geq 2$ can be decomposed into a sum of four extreme operators. In this section, we prove that the same holds when $\dim(\operatorname{Ker} T) \leq 1$.

LEMMA 3.1. *Let T be a positive linear operator on E with $T(E_{11}) = \lambda E_{11}$, $T(E_{22}) = E_{22}$, $T(E_{12}) = E_{12}$, $T(\widetilde{E}_{12}) = 0$ where $\lambda \geq 1$. Then T is a sum of two extreme operators.*

PROOF: Let $\alpha = \frac{1}{2}(\lambda + \sqrt{\lambda^2 - \lambda})$, $\beta = \frac{1}{2\lambda}(\lambda - \sqrt{\lambda^2 - \lambda})$, then we have $\alpha\beta = \frac{1}{4}$, $(\lambda - \alpha)(1 - \beta) = \frac{1}{4}$, $0 < \alpha < \lambda$, and $0 < \beta < \frac{1}{2}$. We define

$$S_z \begin{pmatrix} a & b+ci \\ b-ci & d \end{pmatrix} = \begin{pmatrix} \alpha a & \sqrt{\alpha\beta}(b+ci) \\ \sqrt{\alpha\beta}(b-ci) & \beta d \end{pmatrix}$$

$$S_w \begin{pmatrix} a & b+ci \\ b-ci & d \end{pmatrix} = \begin{pmatrix} (1-\alpha)a & \sqrt{\alpha\beta}(b+ci) \\ \sqrt{\alpha\beta}(b-ci) & (1-\beta)d \end{pmatrix},$$

then it is clear that $T = S_z + \overline{S}_w$.

LEMMA 3.2. *Let T be a positive linear operator on E with $T(E_{11}) = E_{11}$, $T(E_{22}) = E_{22}$, $T(E_{12}) = cE_{12}$, $T(\widetilde{E}_{12}) = dE_{12}$ where $c^2 + d^2 \neq 0$. Then T is a sum of two extreme operators.*

PROOF: Note that we must have $c^2 + d^2 \leq 1$ since $T \geq 0$. First, we consider the case where $c^2 + d^2 = 1$. Let $c = \cos\tau$, $d = \sin\tau$, and $S = T \circ U_\tau$, then it is routine to verify that $S(E_{ii}) = E_{ii}$, $i = 1, 2$, $S(E_{12}) = E_{12}$, $S(\widetilde{E}_{12}) = 0$. Therefore, S is a sum of two extreme operators by Lemma 3.1 and so is T.

Next, we consider the case with $c^2 + d^2 < 1$. Let $t = 1/\sqrt{c^2 + d^2}$, $\cos\tau = tc$, $\sin\tau = dt$, and $T_1 = tT$ then $T_1(E_{ii}) = tE_{ii}$, $i = 1, 2$, $T_1(E_{12}) = \cos\tau E_{12}$, $T_1(\widetilde{E}_{12}) = \sin\tau E_{12}$. If $S = T_1 \circ U_\tau$, then $S(E_{12}) = E_{12}$, $S(\widetilde{E}_{12}) = 0$, $S(E_{ii}) = tE_{ii}$, $i = 1, 2$. Let $\vec{z}^T = (\sqrt{t}, 1/\sqrt{t})$, $S_1 = S_z \circ S$, then $S_1(E_{11}) = \lambda E_{11}$, $S_1(E_{22}) = E_{22}$,

$S_1(E_{12}) = E_{12}$, $S_1(\widetilde{E}_{12}) = 0$ with $\lambda = t^2$. Therefore, S_1 is a sum of two extreme operators by Lemma 3.1 and hence T is a sum of two extreme positive operators by Corollary 1.3.

LEMMA 3.3. *Let T be a positive linear operator on E with $\dim(\operatorname{Ker} T) \leq 1$. If $T(E_{ii}) = E_{ii}$, $i = 1, 2$, $T(E_{12}) = cE_{12}$, $T(\widetilde{E}_{12}) = d\widetilde{E}_{12}$ where $0 \leq |d| \leq c \leq 1$, $d \in \mathbb{R}$, then T is a sum of two extreme operators.*

PROOF: Note that if $c = 1$, then $T = dI + (1-d)\overline{I}$, a sum of two extreme operators. Thus, we assume $c < 1$. Let $\alpha = \frac{1}{2}(1 - cd - \sqrt{(1-cd)^2 - (c-d)^2})$, $\beta = \frac{1}{2}(1 - cd + \sqrt{(1-cd)^2 - (c-d)^2})$, $\gamma = \frac{c-d}{2}$. Note that $(1 - cd) > (c - d) \geq 0$ from $(1-c)(1+d) > 0$ and that $0 \leq \alpha$, $\beta < 1$, $\alpha\beta = \gamma^2$, $(1-\alpha)(1-\beta) = (c-\gamma)(d-\gamma)$. We define

$$S(E_{11}) = \alpha E_{11},\ S(E_{22}) = \beta E_{22},\ S(E_{12}) = \gamma E_{12},$$
$$S(\widetilde{E}_{12}) = -\gamma \widetilde{E}_{12},\ R(E_{11}) = (1-\alpha)E_{11},$$
$$R(E_{22}) = (1-\beta)E_{22},\ R(E_{12}) = (c-\gamma)E_{12},\ R(\widetilde{E}_{12}) = (d+\gamma)\widetilde{E}_{12}.$$

Then we have

$$S\begin{pmatrix} 1 & re^{i\theta} \\ re^{-i\theta} & r^2 \end{pmatrix} = \begin{pmatrix} \alpha & \gamma r(\cos\theta - i\sin\theta) \\ \gamma r(\cos\theta + i\sin\theta) & \beta r^2 \end{pmatrix}$$
$$R\begin{pmatrix} 1 & re^{i\theta} \\ re^{-i\theta} & r^2 \end{pmatrix} = \begin{bmatrix} 1-\alpha & (c-\gamma)r(\cos\theta + i\sin\theta) \\ (c-\gamma)r(\cos\theta - i\sin\theta) & (1-\beta)r^2 \end{bmatrix}$$

for all $r \geq 0$, $\theta \in \mathbb{R}$. Now, note that S and R are of the form S_z from $\alpha\beta = \gamma^2$, $(1-\alpha)(1-\beta) = (c-\gamma)^2$. Therefore, $T = R + S$ is a sum of two extreme operators.

THEOREM 3.4. *Let T be a positive linear operator on E with $\dim(\operatorname{Ker} T) \leq 1$. If $\dim F \geq 2$ where $F = \operatorname{Span}\{\vec{x}\vec{x}^* \mid T(\vec{x}\vec{x}^*)$ is extreme$\}$, then T is a sum of two extreme operators.*

PROOF: Let $\{\vec{x}\vec{x}^*, \vec{y}\vec{y}^*\}$ be a basis of F with $\vec{x}^*\vec{x} = \vec{y}^*\vec{y} = 1$ and let $T(\vec{x}\vec{x}^*) = \vec{z}\vec{z}^*$, $T(\vec{y}\vec{y}^*) = \vec{w}\vec{w}^*$. Note that $\{\vec{z}\vec{z}^*, \vec{w}\vec{w}^*\}$ is linearly independent since otherwise $\dim(T(E)) \le 1$, i.e. $\dim(\operatorname{Ker} T) \ge 3$. We apply Lemma 2.1 to find S_Q and S_R such that $S_Q(E_{11}) = \vec{x}\vec{x}^*$, $S_Q(E_{22}) = \vec{y}\vec{y}^*$, $S_R(\vec{z}\vec{z}^*) = E_{11}$, $S_R(\vec{w}\vec{w}^*) = E_{22}$, and let $S = S_R \circ T \circ S_Q$. Then we have $S(E_{ii}) = E_{ii}$, $i = 1, 2$ and hence $S(E_{12})$, $S(\widetilde{E}_{12}) \in \operatorname{Span}\{E_{12}, \widetilde{E}_{12}\}$ from $S \ge 0$.

Let $S(E_{12}) = aE_{12} + b\widetilde{E}_{12}$, $S(\widetilde{E}_{12}) = cE_{12} + d\widetilde{E}_{12}$, where a, b, c, $d \in \mathbb{R}$ and let $\tan 2\tau = -2(ac + bd)/(c^2 + d^2 - a^2 - b^2)$, $S_1 = S \circ U_\tau$. Then

$$\begin{aligned}&a\cos\tau + c\sin\tau + i(b\cos\tau + d\sin\tau)\\ =&i\lambda(-a\sin\tau + c\cos\tau + i(-b\sin\tau + d\cos\tau))\end{aligned}$$

for some real λ. Note that the left hand side of the above is the $(1,2)$-component of $S_1(E_{12})$ and the right hand side is $i\lambda$ times the $(1,2)$-component of $S_1(\widetilde{E}_{12})$. Therefore, there exists σ such that $S_2 = S_\sigma \circ S_1 = S_\sigma \circ S \circ U_\tau$ satisfies $S_2(E_{12}) = \alpha E_{12}$, $S_2(\widetilde{E}_{12}) = \beta\widetilde{E}_{12}$ with α, $\beta \in \mathbb{R}$. Note that we could take $\tau = \sigma = 0$ when $a = b = c = d$, and $\tau = \frac{\pi}{4}$ when $c^2 + d^2 = a^2 + b^2$. We may assume $|\alpha| \ge |\beta|$ by applying $U_{\frac{\pi}{2}}$ if necessary and also assume $\alpha > 0$ by applying U_π. Now, note that we cannot have $\alpha > 1$ since $S_2 \ge 0$ and hence we have $0 \le |\beta| \le \alpha \le 1$. Therefore, by Lemma 3.3, S_2 is a sum of two extreme operators and hence so is T by Corollary 1.3.

EXAMPLE 3.5. Let $T(E_{ii}) = \sqrt{2}E_{ii}$, $i = 1, 2$, $T(E_{12}) = E_{12}$, $T(\widetilde{E}_{12}) = E_{12}$. Then we clearly have $T \ge 0$ and $\dim F \ge 2$. As in the proof of Theorem 3.4, we take $\tau = \frac{\pi}{4}$ since we have $c^2+d^2-a^2-b^2 = 0$ in this case. Let $S = T \circ U_\tau$, then $S(E_{ii}) = \sqrt{2}E_{ii}$, $i = 1, 2$, $S(E_{12}) = \sqrt{2}E_{12}$, $S(\widetilde{E}_{12}) = 0$. Thus, we can write $S = \frac{1}{\sqrt{2}}I + \frac{1}{\sqrt{2}}\overline{I}$ where I is the identity operator. Therefore $T = \frac{1}{\sqrt{2}}I \circ U_{-\tau} + \frac{1}{\sqrt{2}}\overline{I} \circ U_{-\tau}$, i.e. T is a sum of two extreme operators.

LEMMA 3.6. *Let T be a positive linear operator on E with $T(E_{11}) = 0$. Then T is a sum of two extreme operators.*

PROOF: Note that $T(E_{12}) = T(\widetilde{E}_{12}) = 0$ since $T \ge 0$ and hence we have $\dim(T(E)) = 1$. Let $T(E_{22}) = P$ where $P \ge 0$ and let UPU^*

be diagonal with a unitary matrix U. If $S = U \circ T$, then $S(E_{22}) = d_1E_{11} + d_2E_{22}$ for some d_1, $d_2 \geq 0$. Thus, we have $S = V \circ S_{\vec{z}} + S_{\vec{z}}$ where $\vec{z} = \vec{e}_2$, $V = E_{12}$ and hence T is a sum of two extreme operators.

LEMMA 3.7. *Let T be a positive linear operator on E and $F = \operatorname{Span}\{\vec{x}\vec{x}^* \mid T(\vec{x}\vec{x}^*)$ is extreme$\}$. If $\dim F \geq 3$, then T is a sum of two extreme operators.*

PROOF: In view of Lemma 2.1 and Corollary 1.3, we may assume $T(E_{11}) = E_{11}$, $T(E_{22}) = E_{22}$, $T(E_{12}) = cE_{12}$, $T(\widetilde{E}_{12}) = \begin{pmatrix} 0 & ce^{i\tau} \\ ce^{-i\tau} & 0 \end{pmatrix}$ with $c > 0$. Here we have applied a unitary map of the form S_α in front of T to obtain the form in $T(E_{12})$ and $T(\widetilde{E}_{12})$. Note that we have $c^2(1 + |\cos\tau|) \leq 1$ since $T \geq 0$. Now, from $\dim F \geq 3$, we have

$$T\begin{pmatrix} 1 & re^{i\theta} \\ re^{-i\theta} & r^2 \end{pmatrix} = \begin{pmatrix} 1 & rc(\cos\theta + e^{i\tau}\sin\theta) \\ rc(\cos\theta + e^{-i\tau}\sin\theta) & r^2 \end{pmatrix}$$

is extreme for some $r \neq 0$. Therefore, $c^2(1+\sin 2\theta\,\cos\tau) = 1$ for some $\theta \in \mathbb{R}$, which implies $c^2(1 + |\cos\tau|) = 1$.

Note that T is a sum of two extreme operators if and only if so is $\overline{T}$ and hence we may assume $0 \leq \tau \leq \pi$. First, we consider the case $0 \leq \tau \leq \frac{\pi}{2}$ so that $\cos\tau \geq 0$.

Let

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ e^{-\frac{\pi}{4}i} & -e^{-\frac{\pi}{4}i} \end{pmatrix}, \qquad V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{\tau/2i} \\ 1 & -e^{\tau/2i} \end{pmatrix},$$

and $S = V \circ T \circ U$. Then $S(E_{ii}) = E_{ii}$, $i = 1,2$, $S(E_{12}) = E_{12}$, $S(\widetilde{E}_{12}) = t\widetilde{E}_{12}$ where $t = \tan(\tau/2)$. Note that $0 \leq t \leq 1$. If $t = 1$, then S is extreme and if $t < 1$ then $S = \lambda I + (1-\lambda)\overline{I}$ where $\lambda = (1+t)/2$. Therefore, S is a sum of two extreme operators and hence so is T.

Next, we consider the case with $\frac{\pi}{2} \leq \tau \leq \pi$. We repeat the same process with

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ e^{\frac{\pi}{4}i} & -e^{\frac{\pi}{4}i} \end{pmatrix}, \qquad V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -ie^{\frac{\tau}{2}i} \\ 1 & ie^{\frac{\tau}{2}i} \end{pmatrix}$$

to obtain $S(E_{ii}) = E_{ii}$, $i = 1, 2$, $S(E_{12}) = E_{12}$, $S(\widetilde{E}_{12}) = \cot(\tau/2)\widetilde{E}_{12}$. Thus, by a similar argument, S is a sum of two extreme operators and hence so is T.

LEMMA 3.8. *Let T be a positive linear operator on E with $T(E_{11}) = E_{11}$, $T(E_{22}) = dE_{11} + b^2 E_{22}$, $T(E_{12}) = bE_{12}$, $T(\widetilde{E}_{12}) = aE_{11} + c\widetilde{E}_{12}$ where $d > 0$, $b \geq c \geq 0$. Then T is a sum of three extreme operators.*

PROOF: From $T \geq 0$, we have

$$T\begin{pmatrix} 1 & re^{i\theta} \\ re^{-i\theta} & r^2 \end{pmatrix} = \begin{pmatrix} 1 + dr^2 + ar\sin\theta & br\cos\theta + icr\sin\theta \\ br\cos\theta - icr\sin\theta & b^2r^2 \end{pmatrix} \geq 0$$

for all $r \geq 0$, $\theta \in \mathbb{R}$. Therefore, we must have

$$1 + dr^2 + ar\sin\theta \geq \cos^2\theta + \gamma^2\sin^2\theta \quad \text{with} \quad \gamma = c/b$$

and hence

$$\left(r + \frac{a}{2d}\sin\theta\right)^2 + \left(\frac{1-\gamma^2}{d} - \frac{a^2}{4d^2}\right)\sin^2\theta \geq 0$$

for all $r \geq 0$, $\theta \in \mathbb{R}$. Thus, we obtain $1 - \gamma^2 \geq a^2/4d$. When $1 - \gamma^2 = a^2/4d$, it is clear that $F = \text{Span}\{\vec{x}\vec{x}^* \mid T(\vec{x}\vec{x}^*) \text{ is extreme}\}$ has dimension at least 3. By Lemma 3.7, T is a sum of two extreme operators in this case.

Next, we consider the case with $1 - \gamma^2 > a^2/4d$. Let $\alpha = d - a^2/(4(1-\gamma^2))$, $S = \alpha V \circ S_z$, $z = e_2$, $V = E_{12}$. Then it is routine to verify that $T_1 = T - S$ is positive and the corresponding F has dimension at least 3. Therefore, Lemma 3.7 applies so that T_1 is a sum of two extreme operators. Thus, T is a sum of three extreme operators.

LEMMA 3.9. *Let T be a positive linear operator on E with $T(E_{11}) = E_{11}$, $T(E_{22}) = d_1E_{11} + d_2E_{22}$, $T(E_{12}) = a_1E_{11} + bE_{12}$, $T(\widetilde{E}_{12}) = a_2E_{11} + c\widetilde{E}_{12}$ where $b^2 \geq c^2$. Let $\vec{x}_n = \frac{1}{\sqrt{1+r_n^2}}(1, re^{-i\alpha_n})^T$, $r_n > 0$, $r_n \to 0$ and $\lambda_n = \max\{\lambda > 0 \mid \lambda\vec{x}_n\vec{x}_n^* \leq T(\vec{x}_n\vec{x}_n^*)\}$. If $\dim F = 1$*

where $F = \text{Span}\{\vec{x}\vec{x}^* \mid T(\vec{x}\vec{x}^*) \text{ is extreme}\}$ *and if* $\lambda_n \to 0$ *then* $d_2 = b^2$ and $a_1 = 0$.

PROOF: Note that

$$T(\vec{x}_n\vec{x}_n^*) - \lambda_n\vec{x}_n\vec{x}_n^* =$$
$$\begin{bmatrix} 1 - \lambda_n + d_1 r_n^2 + k\sin(\alpha_n + \theta_0)r_n & r_n(b\cos\alpha_n + ic\sin\alpha_n) - \lambda_n e^{i\alpha_n} \\ r_n(b\cos\alpha_n - ic\sin\alpha_n) - \lambda_n e^{-i\alpha_n} & (d_2 - \lambda_n)r_n^2 \end{bmatrix}$$

is positive and whose determinant is zero for all n by the choice of λ_n. Thus, we obtain $\lambda_n g(r_n, \alpha_n) = f(r_n, \alpha_n)$, where

$$f(r_n, \alpha_n) = d_2(1 + d_1 r_n^2 + k\sin(\alpha_n + \theta_0)r_n) - b^2\cos^2\alpha_n - c^2\sin^2\alpha_n,$$
$$g(r_n, \alpha_n) = 1 + d_2 + d_1 r_n^2 + k\sin(\alpha + \theta_0)r_n - 2b\cos^2\alpha_n - 2c\sin^2\alpha_n,$$

where $k = \sqrt{a_1^2 + a_2^2}$, $\sin\theta_0 = a_1/k$, $\cos\theta_0 = a_2/k$. From $T \geq 0$, we have $f(r, \alpha) > 0$ for all $r > 0$, $\alpha \in \mathbb{R}$ and hence $g(r_n, \alpha_n)$ is also positive for all n. Since $\lambda_n \to 0$ and $g(r_n, \alpha_n)$ is bounded above, we must have $f(r_n, \alpha_n) \to 0$. Now, from $f(r_n, \alpha_n) = d_2 - b^2 + (b^2 - c^2)\sin^2\alpha_n + h(r_n, \alpha_n)r_n$, we must have $d_2 = b^2$ and $\sin\alpha_n \to 0$. Note that $b^2 \neq c^2$ since otherwise $f(r, \alpha) < 0$ for $r > 0$, $\alpha \in \mathbb{R}$. By taking a subsequence, we may assume $\alpha_n \to 0$ or $\alpha_n \to \pi$. We assume $\alpha_n \to 0$ since the case with $\alpha_n = \pi$ can be proved in exactly the same way.

Now, from $f(r, \alpha) = d_2 d_1 r^2 + d_2 k\sin(\alpha + \theta_0)r + (b^2 - c^2)\sin^2\alpha > 0$ for all $r > 0$, $\alpha \in \mathbb{R}$ we must have $\sin\theta_0 = 0$, i.e., $a_1 = 0$.

THEOREM 3.10. *Let* T *be a positive linear operator on* E *with* $\dim(\text{Ker}\, T) \leq 1$. *If* $\dim F = 1$ *where* $F = \text{Span}\{\vec{x}\vec{x}^* \mid T(\vec{x}\vec{x}^*) \text{ is extreme}\}$, *then* T *is a sum of three extreme operators.*

PROOF: By applying Lemma 2.1 and by applying a map of the form S_A, we may assume $T(E_{11}) = E_{11}$, $T(E_{22}) = d_1 E_{11} + d_2 E_{22}$ where d_1, $d_2 > 0$. Note that $T(\vec{x}\vec{x}^*) \neq 0$ for all $\vec{x} \neq 0$ since otherwise $\dim(T(E)) \leq 1$.

We apply unitary maps of the form S_σ so that $S = S_\tau \circ T \circ S_\sigma$ satisfies $S(E_{11}) = E_{11}$, $S(E_{22}) = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, $S(E_{12}) = \begin{pmatrix} a_1 & b \\ b & 0 \end{pmatrix}$,

$S(\widetilde{E}_{12}) = \begin{pmatrix} a_2 & ci \\ -ci & 0 \end{pmatrix}$ where we have $b^2 \geq c^2$ by the choice of τ. We define $\lambda_x = \max\{\lambda \geq 0 \mid \lambda \vec{x}\vec{x}^* \leq T(\vec{x}\vec{x}^*)\}$. Then $\lambda_x \neq 0$ for all $\vec{x} \neq 0$. Let $\lambda_0 = \min\{\lambda_x \mid \vec{x} \in \mathbb{C}^2, \vec{x}^*\vec{x} = 1\}$.

First, consider the case of $\lambda_0 = 0$ and find $\vec{x}_n = \frac{1}{\sqrt{1+r_n^2}}(1, r_n e^{-i\alpha_n})^T$ such that the corresponding λ_n approaches to 0. Note that $r_n \to 0$ in this case since $S(\vec{x}_n\vec{x}_n^*) - \lambda_n \vec{x}_n\vec{x}_n^*$ are all extreme and $\dim F = 1$. Thus, we may apply Lemma 3.9 to conclude $d_2 = b^2$ and $a_1 = 0$ and hence S is a sum of three extreme operators by Lemma 3.8. Therefore, T is a sum of three extreme operators.

Next, we consider the case with $\lambda_0 \neq 0$. It is clear that $T \geq \lambda_0 I$ where I is the identity operator. Let $S = T - \lambda_0 I$. Then $S(E_{11}) = (1 - \lambda_0)E_{11}$ and $S(\vec{x}\vec{x}_0^*)$ is extreme for some $\vec{x}_0 \in \mathbb{C}^2$. If $\vec{x}_0\vec{x}_0^* \neq E_{11}$, then S is a sum of two extreme operators by Theorem 3.4 and hence so is T. If $\vec{x}_0\vec{x}_0^* = E_{11}$, then we must have $\lambda_0 = 1$ and hence $S(E_{11}) = 0$, from which we obtain $\dim(\operatorname{Ker} S) \geq 3$ with E_{12}, $\widetilde{E}_{12} \in \operatorname{Ker} S$. Therefore, $T = \lambda_0 I + S$ and hence T is a sum of three extreme operators by Lemma 3.6.

EXAMPLE 3.11. Let $T(E_{11}) = 2E_{11}$, $T(E_{22}) = I$, $T(E_{12}) = E_{11} + E_{12}$, $T(\widetilde{E}_{12}) = E_{11} + \widetilde{E}_{12}$. Then it is routine to verify that $\dim F = 1$ and $\lambda_0 = 1$ where λ_0 is as defined in Theorem 3.10. Let $S = T - I$, then $S(E_{11}) = S(E_{22}) = S(E_{12}) = S(\widetilde{E}_{12}) = E_{11}$. By Lemma 3.3, S must be a sum of two extreme operators. In fact, we can verify that $S(A) = \langle P, A \rangle = \operatorname{Trace}(PA)$ where $P = \begin{pmatrix} 1 & \frac{1+i}{2} \\ \frac{1-i}{2} & 1 \end{pmatrix}$. The eigenvalues of P are $1 + \frac{1}{\sqrt{2}}$, $1 - \frac{1}{\sqrt{2}}$ and the corresponding eigenvectors are $\left(\frac{1+i}{2}, \frac{1}{\sqrt{2}}\right)^T$, $\left(\frac{1+i}{2}, -\frac{1}{\sqrt{2}}\right)^T$.

Let $U = \begin{pmatrix} \frac{1-i}{2} & \frac{1}{\sqrt{2}} \\ \frac{1-i}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix}$. Then

$$U \begin{pmatrix} a & b + ci \\ b - ci & d \end{pmatrix} U^* = \begin{pmatrix} \frac{a+d}{2} + \frac{b+c}{\sqrt{2}} & \frac{a-d}{2} + \frac{b-c}{\sqrt{2}} i \\ \frac{a-d}{2} - \frac{b-c}{\sqrt{2}} i & \frac{a+d}{2} - \frac{b+c}{\sqrt{2}} \end{pmatrix}.$$

Thus, we have $S = \left(1 + \frac{1}{\sqrt{2}}\right) S_{\vec{z}} \circ U + \left(1 - \frac{1}{\sqrt{2}}\right) V \circ S_{\vec{w}} \circ U$ where

$\vec{z} = \vec{e}_1$, $\vec{w} = \vec{e}_2$, $V = E_{12}$. Therfore, T is a sum of three extreme operators.

LEMMA 3.12. *Let T be a positive linear operator on E. If $T(\vec{x}\vec{x}^*)$ is positive definite for all $\vec{x} \neq 0$, then $T \geq \lambda_0 I$ for some $\lambda_0 > 0$ such that $(T - \lambda_0 I)(\vec{x}_0\vec{x}_0^*)$ is extreme for some $\vec{x}_0 \in \mathbb{C}^2$ with $\vec{x}_0^*\vec{x}_0 = 1$.*

PROOF: For $\vec{x} \neq 0$, we define $\lambda_x = \min\{\lambda > 0 \mid \lambda$ is an eigenvalue of $T(\vec{x}\vec{x}^*)\}$ and let $\lambda_0 = \min\{\lambda_x \mid \vec{x} \in \mathbb{C}^2, \vec{x}^*\vec{x} = 1\}$. We claim that $\lambda_0 \neq 0$. To prove the claim, suppose $\lambda_0 = 0$ and let $\vec{x}_n$ be such that $\vec{x}_n^*\vec{x}_n = 1$ and the corresponding eigenvalues λ_n satisfy $\lambda_n < \frac{1}{n}$. By taking a subsequence when necessary, we may assume $\vec{x}_n \to \vec{x}_0$ for some $\vec{x}_0 \in \mathbb{C}^2$ with $\vec{x}_0^*\vec{x}_0 = 1$. Let $\vec{z}_n$ be the corresponding unit eigenvector of $T(\vec{x}_n\vec{x}_n^*)$. Again, we assume $\vec{z}_n \to \vec{z}_0$ by taking a subsequence. Now, note that $\lambda_n = \vec{z}_n^*T(\vec{x}_n\vec{x}_n^*)\vec{z}_n \to \vec{z}_0^*T(\vec{x}_0\vec{x}_0^*)\vec{z}_0$, i.e. $\vec{z}_0^*T(\vec{x}_0\vec{x}_0^*)\vec{z}_0 = 0$ with $\vec{z}_0^*\vec{z}_0 = 1$. Therefore, we must have 0 is an eigenvalue of $T(\vec{x}_0\vec{x}_0^*)$, which is a contradiction, and the claim is proved. Finally, note that $T(\vec{x}\vec{x}^*) \geq \lambda_x\vec{x}\vec{x}^* \geq \lambda_0\vec{x}\vec{x}^*$ for all $\vec{x} \in \mathbb{C}^2$ and hence $T \geq \lambda_0 I$.

THEOREM 3.13. *Let T be a positive linear operator on E such that $T(\vec{x}\vec{x}^*)$ is positive definite for all $\vec{x} \neq 0$. Then T is a sum of four extreme operators.*

PROOF: By Lemma 3.12, there exists $\lambda_0 > 0$ such that $T \geq \lambda_0 I$ and $(T - \lambda_0 I)(\vec{x}_0\vec{x}_0^*) = \alpha\vec{\xi}\vec{\xi}^*$ for some $\vec{x}_0^*\vec{x}_0 = \vec{\xi}^*\vec{\xi} = 1$. Note that $\alpha \neq 0$ since $T(\vec{x}_0\vec{x}_0^*)$ is positive definite and the $\operatorname{Ker} S \neq \{0\}$ where $S = T - \lambda_0 I$. Now, if $\dim F = 1$ where $F = \operatorname{Span}\{\vec{x}\vec{x}^* \mid S(\vec{x}\vec{x}^*)$ is extreme$\}$ then we apply Theorem 3.10 so that S is a sum of three extreme operators. If $\dim F \geq 2$, then we can apply Theorem 3.4 to conclude S is a sum of two extreme operatrors. Therefore, T is a sum of four extreme operators in any case.

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