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Chain Recurrence in Persistent Dynamical Systems

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ABSTRACT. The purpose of this paper is to study the chain recurrent sets under persistent dynamical systems, and give a necessary condition for a persistent dynamical system to be topologically stable. Moreover, we show that the various recurrent sets depend continuously on persistent dynamical system.

The abstract theory of dynamical systems distinguished various recurrence properties such as periodicity, Poisson stability, nonwanderingness, chain recurrence, etc. The weakest property among them is the property of a point to be chain recurrent.

In [3], Hurley analysed the chain recurrent sets under topologically stable dynamical systems, and Lewowicz [5] introduced the concept of persistence of a dynamical system which is weaker than that of topological stability.

The purpose of this paper is to study the chain recurrent sets under persistent dynamical systems, and give a necessary condition for a persistent dynamical system to be topologically stable. I.U. Bronstein and A.YA. Kopanskii [1] introduced the concepts of weakly nonwandering set and chain recurrent set for a disperse dynamical system (or a dynamical system without uniqueness) and said that, in general, it remains unknown whether or not the weakly nonwandering set is equal to the chain recurrent set. We claim that for a dynamical system (with uniqueness) the weakly nonwandering set is properly contained in the chain recurrence set. Moreover, Ombach [7] showed that the various recurrence mappings (such as α, ω, Ω, CR, etc.) are continuous at f if the system f has the P.O.T.P. (pseudo orbit tracing property) and is expansive. Finally we prove that the various recurrence mappings are also continuous at f if the system f is persistent.

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We consider homeomorphisms acting on a compact metric space. Unless otherwise mentioned, we let X denote a compact metric space with a metric d. Let $H(X)$ denote the collection of all homeomorphisms of X to itself topologized by the C^0-metric

$$d_0(f,g) = \sup\{d(f(x), g(x)) : x \in X\}.$$

A homeomorphism f on X is said to be *topologically stable* if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $d_0(f,g) < \delta$ then there exists a continuous map $h : X \to X$ satisfying $hg = fg$ and $d_0(h, 1_X) < \varepsilon$, where 1_X is the identity map on X. The map h is called the *semiconjugacy* from g to f. We say that $f \in H(X)$ is *expansive* if there exists $e(f) > 0$ such that if $d(f^n(x), f^n(y)) \leq e$ for every $n \in Z$, then $x = y$. Such numbers $e(f)$ are called *expansive constants*. A homeomorphism f on X is called α(or β)-persistent if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $d_0(f,g) < \delta$ and $x \in X$, then there is $y \in X$ satisfying

$$d(f^n(y), g^n(x)) < \varepsilon \quad (\text{or } d(f^n(x), g^n(y)) < \varepsilon),$$

respectively, for all $n \in \mathbf{Z}$.

LEMMA 1. *Any topologically stable homeomorphism is $\alpha(\beta)$-persistent.*

PROOF: Let $f \in H(X)$ be topologically stable, and let $\varepsilon > 0$ be arbitrary. Then we can choose $\delta > 0$ such that if $d_0(f,g) < \delta$ then there is a continuous map $h : X \to X$ with $hg = fh$ and $d_0(h, 1_X) < \varepsilon$. For any $x \in X$, choose $y \in X$ with $h(y) = x$. Then we have

$$\begin{aligned} d(f^n(x), g^n(y)) &= d(f^n(h(y)), g^n(y)) \\ &= d(h(g^n(y)), g^n(y)) < \varepsilon, \end{aligned}$$

for all $n \in \mathbf{Z}$. This means that f is β-persistent. For any $x \in X$, by letting $h(x) = y$, we have

$$\begin{aligned} d(f^n(y), g^n(x)) &= d(f^n(h(x)), g^n(x)) \\ &= d(h(g^n(x)), g^n(x)) < \varepsilon, \end{aligned}$$

for all $n \in \mathbf{Z}$, which implies that f is α-persistent.

Throughout this paper, it will be noted that a persistent homeomorphism means $\alpha(\beta)$-persistent homeomorphism. The converse of the aobve lemma does not hold (See [**5**]). The following theorem gives a necessary condition for an α-persistent homeomorphism to be topologically stable.

THEOREM 2. *An α-persistent homeomorphism is topologically stable if it is expansive.*

PROOF: Let $f \in H(X)$ be α-persistent, and let $e(f)$ be an expansive constant for f. Choose $\varepsilon > 0$ satisfying $\varepsilon < e(f)/4$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_0(f,g) < \delta$, then for any $x \in X$, there is $y \in X$ satisfying

$$d(f^n(y), g^n(x)) < \varepsilon/2$$

for all $n \in Z$. Define a map $h : X \to X$ by $h(x) = y$, where y is an element in X chosen by the property of persistence of f as the above. Then the map h is well-defined. In fact, let z be another element in X such that $d(f^n(z), g^n(x)) < \varepsilon/2$ for all $n \in Z$. Then we have $d(f^n(y), f^n(z)) < \varepsilon$ for all $n \in Z$. Since f is expansive, we get $y = z$.

Now we show that the map h is continuous. Put $h(x) = y$ and $h(x') = y'$, and let $\lambda > 0$ be given. Since f is expansive, we can choose N such that if $d(f^n(y), f^n(y')) \leq e(f)$ for all $-N \leq n \leq N$ then $d(y,y') < \lambda$. Suppose not. Then, for each $N \geq 1$, there exists a_N, $b_N \in X$ such that

$$d(f^n(a_N), f^n(b_N)) \leq e(f) \quad \text{and} \quad d(a_N, b_N) \geq \lambda,$$

for all $-N \leq n \leq N$. Consider the sequences $\{a_N\}$ and $\{b_N\}$. Since X is compact, we have $a_N \to a$ and $b_N \to b$ in X. Then we get $d(a,b) \geq \lambda$ and $d(f^n(a), f^n(b)) \leq e(f)$, for all $n \in \mathbf{Z}$. This contradicts to the expansiveness of f. Since $g : X \to X$ is continuous, given N, there exists $\eta > 0$ such that if $d(x,x') < \eta$ then $d(g^n(x), g^n(x')) < e(f)/2$ for all $-N \leq n \leq N$. Consequently we have

$$\begin{aligned} d(f^n(y), f^n(y')) \leq & d(f^n(y), g^n(x)) + d(g^n(x), g^n(x')) \\ & + d(g^n(x'), f^n(y')) < e(f), \end{aligned}$$

if $d(x,x') < \eta$ and $-N \leq n \leq N$. Thus we obtain $d(y,y') = d(h(x),h(x')) < \lambda$. By now, we have shown that for any $\lambda > 0$ there exists $\eta > 0$ such that if $d(x,x') < \eta$ then $d(h(x),h(x')) < \lambda$, i.e. h is continuous.

Moreover, for any $x \in X$, there exists $y \in X$ satisfying $d(f^{n+1}(y), g^{n+1}(x)) < \varepsilon/2$ for all $n \in \mathbf{Z}$. Then we get $h(g(x)) = f(y) = f(h(x))$, by the definition of h. Hence we have $hg = fh$ on X. This completes the proof of theorem.

COROLLARY 3. *An α-persistent homeomorphism is β-persistent if it is expansive.*

We say that two homeomorphisms f and g are ***topologically conjugate*** if there exists $h \in H(X)$ satisfying $hg = fh$. The homeomorphism h is called a ***topological conjugacy*** between f and g.

In the following theorem, we see that the α (or β)-persistence is invariant under a topological conjugacy.

THEOREM 4. *Any homeomorphism which is topologically conjugate to an α(or β)-persistent homeomorphism is α(or β)-persistent, respectively.*

PROOF: Suppose that an α-persistent homeomorphism f is topologically conjugate to a homeomorphism g, and let h be a topological conjugacy between f and g. Let $\varepsilon > 0$ be given, and choose $0 < \varepsilon' < \varepsilon$ such that if $d(a,b) < \varepsilon'$ then $d(h^{-1}(a), h^{-1}(b)) < \varepsilon$ for a, $b \in X$. Since f is α-persistent, given $\varepsilon' > 0$, there exists $\delta' > 0$ such that if $d_0(f, f_0) < \delta'$ then for any $x \in X$ there is $y \in X$ satisfying $d(f^n(y), f_0^n(x)) < \varepsilon'$ for all $n \in Z$. Given $\delta' > 0$, choose $0 < \delta < \delta'$ such that if $d(a,b) < \delta$ then $d(h(a), h(b)) < \delta'$. Let $g_0 \in H(X)$ be such that $d_0(g, g_0) < \delta$, and put $f_0 = hg_0h^{-1}$. Then we obtain

$$d(h(g(x)), h(g_0(x))) = d(f(h(x)), f_0(h(x))) < \delta'$$

for any $x \in X$, and so $d_0(f, f_0) < \delta'$. Since f is α-persistent, given $h(x)$, there exists $h(y) \in X$ such that

$$d(f^n(h(y)), f_0^n(h(x))) = d(h(g^n(y)), h(g_0^n(x))) < \varepsilon'$$

for all $n \in \mathbf{Z}$. Thus we have $d(g^n(y), g_0^n(x)) < \varepsilon$ for all $n \in Z$. This shows that the map g is α-persistent.

Let $f \in H(X)$. A point $x \in X$ is said to be *periodic* if there exists $n \geq 1$ satisfying $f^n(x) = x$. The set of all periodic points of f will be denoted by $\mathrm{Per}(f)$. A point $x \in X$ is called *nonwandering* if for any neighborhood U of x, there exists $n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$. We denote by $\Omega(f)$ the set of all nonwandering points of f. Let x and y be any points in X, and let $\varepsilon > 0$ be an arbitrary number. A finite sequence $\{x_i\}_{i=0}^n$ in X is called an *ε-chain* from x to y provided that

$$x_0 = x, \quad x_n = y \quad \text{and} \quad d(f(x_i), x_{i+1}) < \varepsilon,$$

for $i = 0, 1, \ldots, n-1$. We say that x is *chain equivalent* to y if for any $\varepsilon > 0$ there exist two ε-chains: from x to y, and from y to x. A point $x \in X$ is called *chain recurrent* if it is chain equivalent to itself. We denote by $CR(f)$ the set of all chain recurrent points of f. There is a natural equivalence relation defined on $CR(f) : x \sim y$ if and only if x is chain equivalent to y. Equivalence classes under this equivalence relation are called *chain components* of f.

A basic problem is to determine when a chain recurrent point is approximated by periodic points of f, or more generally to determine if each ε-chain can be approximated by an actual orbit of f. This problem can be done for α-persistent homeomorphisms on topological manifolds. To show this we need a lemma given in [**6**].

LEMMA 6. *Let X be a compact manifold of* $\dim \geq 2$ *with metric d, and let $\varepsilon > 0$ be arbitrary. Then there exists $\delta(\varepsilon) > 0$ such that if $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ is a finite set of points in $X \times X$ satisfying:*

i) *for each $i = 1, \ldots, n$, $d(x_i, y_i) < \delta$; and*
ii) *if $i \neq j$, then $x_i \neq x_j$ and $y_i \neq y_j$;*

then there is $h \in H(X)$ with $d_0(h, 1_X) < \varepsilon$ and $h(x_i) = y_i$ for $i = 1, \ldots, n$.

THEOREM 6. *Let X be a compact manifold with metric d, and let $f \in H(X)$. If f is α-persistent, then the set of all periodic points of f is dense in $CR(f)$.*

PROOF: If X is one-dimensional then the proof is clear. So, we may assume that the dimension of X is larger than 2. Let $\varepsilon > 0$ be arbitrary. Then we select $\delta_1(\varepsilon) > 0$ as in Lemma 5. Since f is α-persistent, there exists $\delta_2(\delta_1) > 0$ such that if $d_0(f, g) < \delta_2$

and $x \in X$, then $d(f^n(y), g^n(x)) < \delta_1$ for some $y \in X$ and all $n \in \mathbf{Z}$. given $\delta_2 > 0$, we choose $\delta_3(\delta_2) > 0$ as in Lemma 5. Let $x \in CR(f)$, and let $\{x_0, \ldots, x_m\}$ be a δ_3-chain from x to x. Then the set $\{(f(x_0), x_1), \ldots, (f(x_{m-1}), x_m)\}$ satisfies the hypothesis of Lemma 5. Thus there exists $h \in H(X)$ such that

$$d_0(h, 1_X) < \delta_2 \quad \text{and} \quad h(f(x_i)) = x_{i+1},$$

for $i = 0, 1, \ldots, m-1$. Put $g = hf$. Then we have $d_0(f, g) < \delta_2$ and $g^m(x) = x$. Hence there is $y \in X$ satisfying $d(g^n(x), f^n(y)) < \delta_1$ for all $n \in Z$. Consider the set $\{(x, y), (g(x), f(y)), \ldots, (g^m(x), f^m(y))\}$ in $X \times X$ satisfying the hypothesis of Lemma 5. Then we have $h' \in H(X)$ such that

$$d_0(h', 1_X) < \varepsilon \quad \text{and} \quad h'(g^i(x)) = f^i(y)$$

for $i = 0, 1, \ldots, m$. In particular, we have

$$f^m(y) = h'(g^m(x)) = h'(x) = y.$$

This means that $B(x, \varepsilon) \cap \mathrm{Per}(f) \neq \emptyset$, and so completes the proof.

In [**3**], Hurley showed that if f is a topologically stable diffeomorphism on a smooth compact Riemannian manifold X, then each chain component of f contains a dense orbit. Moreover, he claimed that if f is topologically stable, X is connected and $CR(f)$ has interior then $CR(f) = X$.

THEOREM 7. *Let X be a compact manifold with metric d and $f \in H(X)$. If f is α-persistent then each chain component of f contains a dense orbit.*

PROOF: If X is one-dimensional this is clear. Hence we may assume $\dim X \geq 2$. Let F be a chain component in $CR(f)$, and let U and V be any nonempty open sets in F. For any $x \in U$ and $y \in V$, we choose $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$ and $B(y, \varepsilon) \subset V$. Since f is α-persistent, there exists $\delta(\varepsilon) > 0$ such that if $d_0(f, g) < \delta$ then $d(f^n(z), g^n(x)) < \varepsilon$ for some $z \in X$ and all $n \in \mathbf{Z}$. given $\delta > 0$, we select $\delta'(\delta)$ as in Lemma 5. Since x, $y \in F$, we can choose a δ'-chain $\{x_0, \ldots, x_m\}$ from x to y. Then the exists $h \in H(X)$ such that

$$d_0(h, 1_X) < \delta \quad \text{and} \quad h(f(x_i)) = x_{i+1}$$

for $i = 0, 1, \ldots, m-1$. If we let $g = hf$, then there is $z \in X$ satisfying $d(f^n(z), g^n(x)) < \varepsilon$ for all $n \in \mathbf{Z}$. In particular, we have $d(x, z) < \varepsilon$ and $d(f^m(z), g^m(x)) < \varepsilon$. Since $g^m(x) = y$, we get $f^m(z) \in f^m(U) \cap V \neq \emptyset$. This implies that F has a dense orbit.

THEOREM 8. *If $CR(f)$ is connected, then X is the chain component of any point in X.*

PROOF: Let $\varepsilon > 0$ be fixed and $x \in X$. Let $F(x, \varepsilon) = \{y \in CR(f) :$ there exists two ε-chains, from x to y and from y to $x\}$.

First we show that $F(x, \varepsilon)$ is open and closed in $CR(f)$. If $y \in F(x, \varepsilon)$, then there exist two ε-chains: $\{x_i\}_{i=1}^m$ from x to y, and $\{y_j\}_{j=1}^n$ from y to x. Since $f(y) \in B(y_1, \varepsilon)$, we can choose $\delta_1 > 0$ such that $B(f(y), \delta_1) \subset B(y_1, \varepsilon)$. If we use the continuity of f, we can select $\delta_2 > 0$ satisfying

$$f(B(y, \delta_2)) \subset B(f(y), \delta_1) \subset B(y_1, \varepsilon).$$

Let $U = B(y, \varepsilon) \cap B(f(x_{m-1}, \varepsilon)$. Then U is an open neighborhood of y contained in $F(x, \varepsilon)$. In fact, for any $z \in U$, the sequence $\{x_0, x_1, \ldots, x_{m-1}, z\}$ is an ε-chain from x to z, and the sequence $\{z, y_1, \ldots, y_n\}$ is an ε-chain from z to z. Thus we have $U \subset F(x, \varepsilon)$. This shows that $F(x, \varepsilon)$ is open in $CR(f)$. To show that $F(x, \varepsilon)$ is closed in $CR(f)$, we choose a sequence $\{x_i\}$ in $F(x, \varepsilon)$ which converges to $x^0 \in X$. Then there exists $n \in \mathbf{N}$ such that $d(x^i, x^0) < \varepsilon/2$ for all $i \geq n$. Since $x^n \in F(x, \varepsilon) \subset CR(f)$, we can choose an $\varepsilon/2$-chain $\{x_0^n, x_1^n, \ldots, x_k^n\}$ from x^n to x^n. Consider the sequence $\{x_0^n, x_1^n, \ldots, x_{k-1}^n, x^0\}$. Then it is an ε-chain from x^0 to x^n. This means that $F(x, \varepsilon)$ is closed in $CR(f)$.

Next we show that X is the chain component of x. Since $CR(f)$ is connected, we have $CR(f) = F(x, \varepsilon)$. Let $F(x)$ be the chain component of x. Since $F(x) = \bigcap_{\varepsilon > 0} F(x, \varepsilon)$, the proof is completed by showing that $CR(f) = X$. So, we suppose that $CR(f) \neq X$. Then there exists $y \in X$ with $y \notin CR(f)$. Let $\omega_f(y) = \{z \in X : f^ni(y) \to z \text{ for some } n_i \to -\infty\}$, and let $z \in \omega_f(y)$. Then we have

$$z \in \omega_f(y) \subset \Omega(f) \subset CR(f).$$

Since $CR(f) = F(x, \varepsilon)$, we can choose two ε-chains: $\{x_i\}_{i=0}^k$ from x to z, and $\{z_i\}_{i=0}^P$ from z to x. Using the continuity of f, we can select $m < 0$ such that $d(f(z), f^{m+1}(y)) < \varepsilon$. Then the sequence

$$\{x_0, x_1, \dots, x_k, f^{m+1}(y), \dots, f^{-1}(y), y\}$$

is an ε-chain from x to y. Similarly we can construct an ε-chain from y to x. Thus we have $y \in F(x, \varepsilon)$. This contradict to the fact that $y \notin CR(f)$, and so completes the proof.

I.U. Bronstein and A.YA. Kopanskii introduced the notions of weakly nonwandering set and chain recurrent set for a disperse dynamical system (or a dynamical system without uniqueness) on a compact metric space, and said that, in general, it remains unknown whether or not the weakly nonwandering set is equal to the chain recurrent set (see Section 6 in [**1**]). Clearly, a dynamical system (with uniqueness) on a compact metric space s also a disperse dynamical system.

Similarly we introduce the concept of weakly nonwandering set of a dynamical system (or a homeomorphism) on a compact metric space.

A point $x \in X$ is called *weakly nonwandering* (or *weakly periodic*) for $f \in H(X)$ if for any $\varepsilon > 0$ there exists $g \in H(X)$ such that $d_0(f, g) < \varepsilon$ and x is nonwandering (or periodic) for g, respectively. The set of all weakly nonwandering (or weakly periodic) points for f will be denoted y $\Omega_w(f)$ (or $P_w(f)$), respectively. It is clear that $\Omega_w(f)$ is closed and $\Omega(f) \subset \Omega_w(f)$.

In the following example, we show that for a dynamical system (or a homeomorphism) $f$on a compact metric space, the set $\Omega_w(f)$ is not equal to $CR(f)$.

EXAMPLE 9: Let X be the subset of $\mathbf{R}^2$ given by $X = S^1 \cup L$, where $S^1 = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$ and $L = \{(x, y) \in \mathbf{R}^2 : -1 < x < 1 \text{ and } y = 0\}$. Then we define a homeomorphism f_1 on S^1 satisfying: $(-1, 0)$ and $(1, 0)$ are fixed points of f_1, and for any $(x, y) \in S^1 - \{(-1, 0), (1, 0)\}$ the first coordinates of $f_1(x, y)$ is larger than x. Also we define a homeomorphism f_2 on L satisfying: for any $(x,y) \in L$ the first coordinates of $f_2(x,y)$ is less than x. Then we can define a homeomorphism f on the compact metric space X such that $f|_{S^1} = f_1$ and $f|_L = f_2$. For the homeomorphism f, we have $CR(f) = X$ and $\Omega_w(f) = \{(-1,0),(1,0)\}$.

THEOREM 10. *For any $f \in H(X)$, the set $\Omega_w(f)$ is contained in $CR(f)$.*

PROOF: Let $x \in \Omega_w(f)$, and let $\varepsilon > 0$ be arbitrary. Then we have $g \in H(X)$ such that $d_0(f,g) < \varepsilon/4$ and $x \in \Omega(g)$. Choose $\delta < \varepsilon/4$ such that if $d(x,y) < \delta$ then $d(f(x),g(x)) < \varepsilon/4$ and $d(g(x),g(y)) < \varepsilon/4$. Since $x \in \Omega(g)$, there exists $n \geq 1$ satisfying $g^n(B(x,\delta)) \cap B(x,\delta) \neq \emptyset$. If $n = 1$ then the sequence $\{x,x\}$ is an ε-chain for f. In fact, if we choose $y \in B(x,\delta)$ with $g(y) \in B(x,\delta)$, then we get

$$d(f(x),x) < d(f(x),g(x)) + d(g(x),g(y)) + d(g(y),x) < \varepsilon.$$

If $n > 1$ then there exists $y \in B(x,\delta)$ with $g^n(y) \in B(x,\delta)$. Then the sequence $\{x, g(y), \dots, g^{n-1}(y), x\}$ is an ε-chain from x to x. This shows that $x \in CR(f)$.

REMARKS 11: In the proof of Theorem 6, we can see that the set $\Omega_w(f)$ is equal to $CR(f)$ if X is a compact manifold and $f \in H(X)$. But, clearly, the set $\Omega_w(f)$ is not equal to $\Omega(f)$ even if X is a compact manifold and $f \in H(X)$.

For any $f \in H(X)$ and any $x \in X$, we let $\alpha_f(x) = \{y \in X : f^{n_i}(x) \to y \text{ for some } n_i \to \infty\}$ and $\omega_f(x) = \{y \in X : f^{n_i} \to y \text{ for some } n_i \to \infty\}$. Then the sets $\alpha(f) = \bigcup_{x\in X} \alpha_f(x)$ and $\omega(f) = \bigcup_{x \in X} \omega_f(x)$ are called the *negative* and *positive limit sects* for f, respectively. Let us consider the metric space $K(X) = \{X \subset X : A \text{ is closed}\}$ with the Hausdorff metric H

$$H(A,B) = \max\{\sup_{a\in A}(d(a,B), \sup_{b\in B} d(A,b)\}.$$

Let α, ω, Ω, Ω_w, CR denote mappings $H(X) \to K(X)$ sending f to $\overline{\alpha(f)}$, $\overline{\omega(f)}$, $\Omega(f)$, $\Omega_w(f)$, $CR(f)$, respectively.

In [7], Ombach proved that the mappings defined as above are continuous at f if f has the P.O.T.P. and is expansive. It is well-known that a homeomorphism which is expansive and possesses the

P.O.T.P. is topologically stable. By Lemma 1, a topologically stable homeomorphism is persistent. Finally, we show that the mapping defined as above are also continuous at f if f is persistent. To show this we need the concept of upper and lower semi-continuity.

Let Y be a topological space. A mapping $F : Y \to K(X)$ is *upper* (or *lower*) *semi-continuous* at $y \in Y$, if for any $\varepsilon > 0$ there exists a neighborhood U of y such that for any $z \in U$ we have

$$F(z) \subset B_\varepsilon(F(y)) \qquad ((\text{or } F(y) \subset B_\varepsilon(F(z))),$$

respectively, where $B_\varepsilon(A) = \{y \in X : d(x,y) < \varepsilon, \text{ for some } x \in A\}$.

LEMMA 12. *A mapping $F : Y \to F(X)$ is continuous at $y \in Y$ if and only if F is upper and lower semi-continuous at y.*

THEOREM 13. *Let X be a compact manifold, and let $f \in H(X)$ be persistent. Then the mappings α, ω, Ω, Ω_w, CR are continuous at f.*

PROOF: Let $\varepsilon > 0$ be arbitrary. Since f is persistent, we can choose $\delta > 0$ such that if $d_0(f,g) < \delta_1$ and $x \in X$, then $d(f^n(x), g^n(y)) < \varepsilon/3$ for some $y \in X$ and all $n \in \mathbf{Z}$. Let $g \in H(X)$ be such that $d_0(f,g) < \delta_1$. Then for any $x \in X$, there exists $y \in X$ satisfying $d(f^n(x), g^n(y)) < \varepsilon/3$ for all $n \in Z$. This means that

$$\omega_f(x) \subset B_{\frac{\varepsilon}{2}}(\omega_g(y)) \subset B_{\frac{\varepsilon}{2}}(\overline{\omega(g)}), \qquad \text{and}$$
$$\alpha_f(x) \subset B_{\frac{\varepsilon}{2}}(\alpha_g(y)) \subset B_{\frac{\varepsilon}{2}}(\overline{\alpha(g)}).$$

Thus we have

$$\overline{\omega(f)} \subset B_\varepsilon(\overline{\omega(g)}) \quad \text{and} \quad \overline{\alpha(f)} \subset B_\varepsilon(\overline{\alpha(g)}.$$

Since the mapping $CR : H(X) \to K(X)$ is upper semi-continuous at f, by Corollary 3 (a) in [**2**], we can choose $\delta_2 > 0$ such that if $d_0(f,g) < \delta_2$ then $CR(g) \subset B_\varepsilon(CR(f))$. Let $\delta = \min(\delta_1, \delta_2)$, and let $g \in H(X)$ be such that $d_0(f,g) < \delta$. Since $CR(f) = \overline{\text{Per}(f)}$ by Theorem 6, we have

$$CR(f) = \overline{\text{Per}(f)} \subset \overline{\omega(f)} \subset B_\varepsilon(\overline{\omega(g)}) \subset B_\varepsilon(CR(g)).$$

This implies that all considered mappings are lower semi-continuous at f. On the other hand, we have

$$\overline{\alpha(g)} \cup \overline{\omega(g)} \subset CR(g) \subset B_\varepsilon(CR(f)) = B_\varepsilon(\overline{\mathrm{Per}(f)}).$$

This means that all considered mappings are upper semi-continuous at f. By Lemma 12, we completes the proof.

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