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A Radon-Nikodym Derivative of the Product Measure

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1.Introduction

The purpose of this note is to sketch how a Radon-Nikodym derivative of the product measure is represented.

To prove the main result, we will introduce some definitions and theorems. Notations and definitions used in this note are collected in [1], [2] and [3].

Theorem 1.1 (Radon-Nikodym theorem). *Let (X, S) be a measurable space and let μ and ν be finite measures on (X, S). If ν is absolutely continuous with respect to μ, then there is an S-measurable function $g : X \longrightarrow [0, \infty)$ such that $\nu(E) = \int_E g d\mu$ holds for each E in S.*

The function g is unique up to μ-almost everywhere equality.

In the Theorem 1.1 an S-measurable function g on X that satisfies $\nu(E) = \int_E g d\mu$ for each E in S is called the Radon-Nikodym derivative of ν with respect to u, which is denoted by $\frac{d\nu}{d\mu}$.

We now consider the product measure.

Definition 1.2: If (X_1, S_1, μ_1) and (X_2, S_2, μ_2) are finite measure spaces, then the set function λ, defined for every set E in $S_1 \times S_2$ by

$$\lambda(E) = \int \mu_2(E_{x_1}) d\mu_1(x_1) = \int \mu_1(E_{x_2}) d\mu_2(x_2),$$

is a finite measure with the property that, for every measurable rectangle $A_1 \times A_2$,

$$\lambda(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2).$$

The measure λ is called the *product* of the given measure μ_1 and μ_2, and is denoted by

$$\lambda = \mu_1 \times \mu_2;$$

the measure space $(X_1 \times X_2, S_1 \times S_2, \mu_1 \times \mu_2)$ is called the *Cartesian product* of the given measure spaces.

Proceeding by mathematical induction we can define the *Cartesian product* of finite measure spaces (X_i, S_i, μ_i), $i = 1, 2, \ldots, n$, there is one and only one measure μ (denoted by $\mu_1 \times \mu_2 \times \cdots \times \mu_n$) on $S \times \cdots \times S$ such that $\mu(A_1 \times A_2 \times \ldots A_n) = \prod_{i=1}^{n} \mu_i(A_i)$ for every measurable rectangle $A_1 \times A_2 \times \cdots \times A_n$.

The classical product measure theorem in [3] extends as follows;

Theorem 1.3. *For each $i = 1, 2, \ldots,$ let (X_i, S_i, μ_i) be an arbitrary probability space. Let $X = \times_{i=1}^{\infty} S_i$, $S = \times_{i=1}^{\infty} S_i$. There is a unique probability measure μ on S such that $\mu\{X \in S : X_1 \in A_1, \ldots, X_n \in A_n\} = \prod_{i=1}^{n} \mu_i(A_i)$ for all $n = 1, 2, \ldots$ and all $A_i \in S_i$, $i = 1, 2, \ldots$*

We call μ the *product* of the μ_i, and write $\mu = \prod_{i=1}^{\infty} \mu_i$.

2. Main result

Now we need the following lemma,

Lemma 2.1. *Let (X_1, S_1) and (X_2, S_2) be measurable spaces, let μ_1 and ν_1 be finite measures on (X_1, S_1), and let μ_2 and ν_2 be finite measures on (X_2, S_2), if ν_1 is absolutely continuous with respect to μ_1 and ν_2 is absolutely continous with respect to μ_2, then $\nu_1 \times \nu_2$ is absolutely continuous with respect to $\mu_1 \times \mu_2$.*

Proof: If $(\nu_1 \times \nu_2)(A_1 \times A_2) = 0$ for any measurable set $A_1 \times A_2 \in S_1 \times S_2$, then $\mu_1(A_1) = 0$ or $\nu_2(A_2) = 0$ for $A_1 \in S_1$ and $A_2 \in S_2$. Hence $\mu_1(A_1) = 0$ or $\mu_2(A_2) = 0$,

$$(\mu_1 \times \mu_2)(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2) = 0$$

Therefore $\nu_1 \times \nu_2$ absolutely continous with respect to $\mu_1 \times \mu_2$.

We write $\nu << \mu$ if ν is absolutely continous with respect to μ.

Theorem 2.2. *Let (X_i, S_i) be measurable spaces, let μ_i and ν_i be finite measures on (X_i, S_i) where $i = 1, 2, \ldots, n$. If $\nu_1 << \mu_1, \nu_2 << \mu_2, \ldots, \nu_n << \mu_n$, then*

$$\nu_1 \times \nu_2 \times \cdots \times \nu_n << \mu_1 \times \mu_2 \times \cdots \times \mu_n.$$

Proof: Clear from the Lemma 2.1.

Now we will show how a Radon-Nikodym derivative of the n-dimensional product measure is represented.

Theorem 2.3. *Let (X_i, S_i) be measurable spaces, let μ_i and ν_i be finite measures on (X_i, S_i) and $\nu_i << \mu_i$, $i = 1, 2, \ldots, n$.*

Then a Radon-Nikodym derivative of the product measure $\nu = \nu_1 \times \nu_2 \times \cdots \times \nu_n$ with respect to $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ is represented by

$$\frac{d(\nu_1 \times \nu_2 \times \cdots \times \nu_n)}{d(\mu_1 \times \mu_2 \times \cdots \times \mu_n)} = \frac{d\nu_1}{d\mu_1} \cdot \frac{d\nu_2}{d\mu_2} \cdots \frac{d\nu_n}{d\mu_n}.$$

Proof: Since $\mu_1 \times \mu_2 \times \cdots \times \mu_n$ and $\nu_1 \times \nu_2 \times \cdots \times \nu_n$ are finite measures on

$$(X_1 \times X_2 \times \cdots \times X_n, S_1 \times S_2 \times \cdots \times S_n)$$

and $\nu_1 \times \nu_2 \times \cdots \times \nu_n << \mu_1 \times \mu_2 \times \cdots \times \mu_n$. By the Radon-Nikodym theorem on a product measure space, there is an $S_1 \times S_2 \times \cdots \times S_n$-measurable function $G : X_1 \times X_2 \times \cdots \times X_n \to [0, \infty)$ such that

$$(\nu_1 \times \nu_2 \times \cdots \times \nu_n)(A_1 \times A_2 \times \cdots \times A_n) = \int_{A_1 \times A_2 \times \cdots \times A_n} G d(\mu_1 \times \mu_2 \times \cdots \times \mu_n)$$

for $A_i \in S_i$, $i = 1, 2, \ldots, n$.

On the other hand, since $\nu_i << \mu_i$ for each i, there is a unique S-measurable function g_i such that $\nu_i(A_i) = \int_{A_i} g_i d\mu_n$ for $A_i \in S_i$, $i = 1, 2, \ldots, n$. Hence

$$\begin{aligned}(\nu_1 \times \nu_2 \times \cdots \times \nu_n)(A_1 \times A_2 \times \cdots \times A_n) &= \nu_1(A_1)\nu_2(A_2)\ldots\nu_n(A_n) \\ &= \int_{A_1} g_1 d\mu_1 \int_{A_2} g_2 d\mu_2 \cdots \int_{A_n} g_n d\mu_n \\ &= \int_{A_1 \times A_2 \times \cdots \times A_n} G d(\mu_1 \times \mu_2 \times \cdots \times \mu_n).\end{aligned}$$

Therefore a Radon-Nikodym derivative of the n-dimensional product measure can be represented by

$$\frac{d(\nu_1 \times \nu_2 \times \cdots \times \mu_n)}{d(\mu_1 \times \mu_2 \times \cdots \times \mu_n)} = \frac{d\nu_1}{d\mu_1} \cdot \frac{d\nu_2}{d\mu_2} \cdots \frac{d\nu_n}{d\mu_n}.$$

An extension of a Radon-Nikodym derivative to an infinite dimensional product measure is followed by Theorem 1.3 and the above.

References

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