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Recent Development of Angular Spectrum Models for Water Wave Propagation
波浪의 變形을 計算하기 위한 角스펙트럼모델의 最近開發

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Abstract□ As an effort for modeling the water waves propagating in a wide range of incident angles as when waves are diffracted behind a breakwater, angular spectrum models have been developed. In this paper, the concept of the angular spectrum is illustrated and the recently developed angular spectrum models are introduced.

要 旨: 防波堤의 뒤에서 波浪이 廻折될 때와 같이 波浪이 여러 方向으로 傳播되는 경우 波浪의 變形을 計算하기 위하여 角스펙트럼 모델이 開發되어왔다. 본 論文에서는 우선 角스펙트럼의 概念을 說明하고, 最近에 이를 土臺로 하여 開發된 角스펙트럼 모델들을 紹介한다.

1. INTRODUCTION

As water waves propagate from the deep ocean to the coast, they are transformed continuously in both space and time by refraction, diffraction, shoaling, and interactions among waves themselves or with tidal or other ambient currents. Numerous means have been developed to predict the transformation of waves including the effects of these phenomena. Until the 1960s the theories for refraction and diffraction for water waves were developed independently. One of the first attempts for describing the combined effects of refraction and diffraction was made by Berkhoff (1972). His equation known as the mild-slope equation is given by

$$\nabla_h \cdot (CC_g \nabla_h \Phi) + k^2 CC_g \Phi = 0 \tag{1}$$

where ∇_h is a gradient operator in horizontal coordinates (x, y), k (x, y) is the local wave number, C (x, y) and C_g (x, y) are the local phase and group velocities, respectively, and Φ (x,y) is a two-dimensional velocity potential which is related to the velocity potential for the wave motion, $\Phi'(x, y, z, t)$, by

$$\Phi' = i\frac{g}{\omega}\Phi\frac{\cosh k(h+z)}{\cosh kh}e^{-i\omega t} \tag{2}$$

where $i=\sqrt{-1}$, g is the gravitational acceleration, ω is the angular frequency, t is time, h (x, y) is the water depth, and the vertical coordinate z is measured vertically upwards from the still water line. The mild-slope equation is essentially of elliptic type, so it can be solved only when a boundary condition along a closed curve is given. Numerical finite-element techniques have been used by Berkhoff (1972) to treat arbitrary boundary problems such as harbors and islands, which need to solve a set of simultaneous equations over the whole area. In order to obtain a numerical solution for short waves over a large model area, a great amount of computing time and storage is thus needed.

Radder (1979) demonstrated a method for obtaining a parabolic equation by applying a splitting mat-

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rix approach to the mild-slope equation. Similar equations including nonlinear effects were derived by Kirby and Dalrymple (1983) and Liu and Tsay (1984). The parabolic equation method using marching solution technique is advantageous for solving the propagation of waves on a beach with a given wave condition at the offshore boundary. Also it is more convenient and rapid than the finite-element models using the elliptic mild-slope equation. There is a significant drawback of the parabolic method, however, in that it requires that the waves propagate nearly along a given direction (say x direction). Deviations of the wave direction, due to refraction and diffraction, away from this direction lead to errors, which are usually small for waves propagating in directions not greater than 30 degrees from the x direction.

Efforts towards developing a large-angle parabolic model have been made. Booij (1981) developed a parabolic model including current effects, improving Radder's model by about 13 degrees in angles of propagation from the x direction. Kirby (1986a) utilized the Padé approximant to find a higher-order correction to the usual lowest order parabolic approximation, giving reasonably accurate results up to about 50 degrees of propagation from the x axis. Kirby (1986b) additionally used the minimax approximation to obtain better accuracy for waves propagating at large angles at the expense of degradation of accuracy for waves incident at small angles.

Recently the Ocean Engineering Group at the University of Delaware, USA, has developed a series of angular spectrum models, which are theoretically valid for angles of propagation up to ±90 degrees from the x direction and permits solution by a marching method like the parabolic method. The purpose of this paper is to illustrate the concept of the angular spectrum and to introduce the recently developed angular spectrum models for solving various wave propagation problems.

2. THE ANGULAR SPECTRUM AND ITS PHYSICAL INTERPRETATION

In order to illustrate the concept of the angular spectrum and its physical significance, we consider the Helmholtz equation in Φ (x, y):

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}+k^2 \Phi=0 \tag{3}$$

which is equivalent to Eq. (1) in water of constant depth.

Suppose that a wave field represented by Φ (0, y) is incident on the line x = 0, propagating into the half-plane x>0. The Fourier transform of Φ (0, y) in the y direction is

$$\hat{\Phi}(0, \lambda)=\int_{-\infty}^{\infty} \Phi(0, y) e^{-i\lambda y} dy \tag{4}$$

where the circumflex denotes a transformed variable and λ is the continuous Fourier parameter. The inverse Fourier transform is

$$\Phi(0, y)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\Phi}(0, \lambda) e^{i\lambda y} d\lambda \tag{5}$$

Noting that the unit-amplitude plane wave propagating in the direction of $\vec{k}=(\sqrt{k^2-\lambda^2}, \lambda)$ is $\exp[i(\sqrt{k^2-\lambda^2}x+\lambda y)]$, exp (i λ y) may be regarded as a unit-amplitude plane wave propagating in that direction at x = 0. The complex amplitude of that plane wave component is simply $(1/2\pi)\ \hat{\Phi}(0, \lambda)$ d as can be seen in Eq.(5). For this reason, $\hat{\Phi}$ (0, λ) is called the angular spectrum of the wave field Φ (0, y). Simply the angular spectrum is nothing but the Fourier transform of a wave field along a straight line, each component of which represents the complex amplitude of the plane wave propagating in a certain direction.

The Fourier transform of Eq.(3) in the y direction provides an equation for the evolution of the angular spectrum $\hat{\Phi}$ (x, λ):

$$\hat{\Phi}_{xx}+(k^2-\lambda^2)\hat{\Phi}=0 \tag{6}$$

where subscripts denote partial differentiation. An elementary solution to this equation for constant k is

$$\hat{\Phi}(-\chi, \lambda)=\hat{\Phi}(0, \lambda)\exp(i\sqrt{k^2-\lambda^2}x) \tag{7}$$

This result will be interpreted differently depending

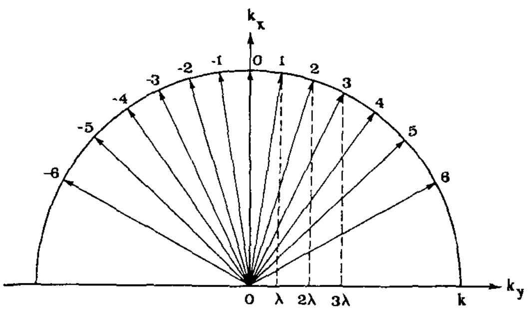

Fig. 1. Diagram of the Fourier decomposition of the wave field on a row with an angular spectrum (with lateral wavenumbers, $s\lambda$, $s=0, \pm1, \pm2,...$). k_x and k_y are the wavenumbers in the x and y directions, respectively.

on the magnitude of $(k^2-\lambda^2)$. If $(k^2-\lambda^2)>0$, then the effect of propagation over a distance x is simply a change in the relative phases of the various components of the angular spectrum. Since each plane-wave component propagates at a different angle, each travels a different distance to reach a given observation point and relative phase delays are thus introduced. If $(k^2-\lambda^2)<0$, these wave components decay exponentially as they propagate in the x direction. Such components of the angular spectrum are called evanescent modes. In water wave problems, these evanescent modes are usually neglected and only the progressive modes at the offshore boundary are carried into the domain, assuming the energy of the evanescent modes is negligibly small compared with that of the progressive modes. The limiting case, $(k^2-\lambda^2)=0$, corresponds to the plane wave propagating in the y direction, contributing no net energy flow in the x direction.

Finally, the inverse Fourier transform of Eq.(7) gives the solution to Eq.(3) in terms of the initial angular spectrum $\hat{\Phi}(0, \lambda)$:

$$\Phi(x, y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\Phi}(0, \lambda)\exp(i\sqrt{k^2-\lambda^2}x)\exp(i\lambda y)d\lambda \qquad (8)$$

This equation implies that it is sufficient to know the free surface displacement on the line $x=0$ to determine it at any point in the half-plane $x>0$.

In the actual computation using discrete data values on a computational grid, a discrete Fourier transform is used under the assumption that the model domain is periodic in the y direction. By discretizing the domain of width l by $N+1$ equidistant points of spacing $\Delta y=l/N$ so that $\Phi(x, 0)=\Phi(x, N\Delta y)$, the velocity potential $\Phi(x, y)$ defined on the first N points can be transformed into discrete Fourier modes by

$$\Phi_s(x)=\frac{1}{N}\sum_{j=0}^{N-1}\Phi(x, j\Delta y)e^{-is\lambda j\Delta y},$$
$$s=0, \pm1, \pm2, \cdots, \pm(\frac{N}{2}-1), -\frac{N}{2} \qquad (9)$$

which describe the wave components propagating in different directions as indicated in Fig. 1. The inversion formula is

$$\Phi(x, j\Delta y)=\sum_s\Phi_s(x)e^{is\lambda j\Delta y},$$
$$j=0, 1, 2, \cdots, (N-1) \qquad (10)$$

where

$$\lambda=\frac{2\pi}{N\Delta y} \qquad (11)$$

which is different from the continuous Fourier parameter λ used previously. These transforms can be performed efficiently by using a fast Fourier transform.

In this section the concept of the angular spectrum was illustrated on the xy plane, so that the angular spectrum is defined on the lines of constant x and its evolution is computed on the half-plane $x>0$. In other branches of physics and engineering (e.g. Ratcliffe 1956; Gabor 1961; Clemmow 1966), the angular spectrum has been used in the xyz space, in which the angular spectrum is defined on the yz plane and its evolution is computed in the half-space $x>0$. The use of the angular spectrum in the three dimensional space is discussed in Goodman (1968) and needs the Fourier transform in both y and z directions.

3. LINEAR MODEL ON A BEACH WITH STRAIGHT ISO-BATHS

In the previous section, the linear angular spectrum model for water waves on constant depth was developed. The next simple case may be when waves propagate on a beach with straight and parallel (in the y direction) bottom contours. For this case the mild-slope equation (1) can be written as

$$(CC_g\Phi_x)_x+CC_g\Phi_{yy}+k^2CC_g\Phi=0 \quad (12)$$

The discrete Fourier transform of the above equation in the y direction leads to the decomposition of the potential into directional modes

$$(CC_g\Phi_{s_x})_x+(k^2-(s\lambda)^2)CC_g\Phi_s=0 \quad (13)$$

Splitting the potential into the forward-propagating and backscattered potentials and neglecting the assumed small backscattered wave, Dalrymple and Kirby (1988) constructed the angular spectrum model for the forward-propagating wave given by

$$2\sqrt{k^2-(s\lambda)^2}CC_g\Phi_{s_x}^+-2i(k^2-(s\lambda)^2)CC_g\Phi_s^+ + [\sqrt{k^2-(s\lambda)^2}CC_g]_x\Phi_s^+=0,$$
$$s=0, \pm1, \pm2, \cdots, \pm(\frac{N}{2}-1), -\frac{N}{2} \quad (14)$$

in which the superscript + denotes the forward-propagating wave. This equation can be solved analytically to give the solution

$$\Phi_s^+(x)=\Phi_s^+(0)[\frac{(CC_g)_o\sqrt{k_o^2-(s\lambda)^2}}{CC_g\sqrt{k^2-(s\lambda)^2}}]^{1/2}$$
$$\exp(i\int^x\sqrt{k^2-(s\lambda)^2}dx) \quad (15)$$

in which the subscript o indicates initial conditions at $x=0$. The bracketed term contains the shoaling and refraction coefficients associated with gradual water depth changes. In water of constant depth this equation reduces to Eq. (7). Note that on constant depth the magnitude of each component of the angular spectrum remains constant and only its relative phase changes with x.

4. QUASI-NONLINEAR MODEL ON IRREGULAR BATHYMETRY

Dalrymple and Kirby's (1988) model was extended to the case of irregular bathymetry by Dalrymple, Suh, Kirby and Chae (1989). The governing equation is again taken to be the mild-slope equation (1). Using the definition of $p(x, y)=CC_g$ and $\phi=\sqrt{p}\Phi$ as in Radder (1979), the mild-slope equation becomes an Helmholtz equation

$$\nabla_h^2\phi+k_c^2\phi=0 \quad (16)$$

in which

$$k_c^2=k^2-\frac{\nabla_h^2\sqrt{p}}{\sqrt{p}} \quad (17)$$

Defining a laterally-averaged wavenumber, $\bar{k}$, as

$$\bar{k}^2=\frac{1}{\ell}\int_0^\ell k_c^2dy \quad (18)$$

so that

$$k_c^2=\bar{k}^2(1-\nu^2) \quad (19)$$

in which

$$\nu^2=1-\frac{k_c^2}{\bar{k}^2} \quad (20)$$

Eq. (16) becomes

$$\nabla_h^2\phi+\bar{k}^2\phi-\bar{k}^2\nu^2\phi=0 \quad (21)$$

Note that $\bar{k}^2$ is a function of x only and the variability of depth in the y direction is contained in $\nu^2(x, y)$. The Fourier transform of the above equation leads to the equations for directional modes

$$\phi_{s_{xx}}+(\bar{k}^2-(s\lambda)^2)\phi_s-\bar{k}^2F_s(\nu^2\phi)=0 \quad (22)$$

in which F_s denotes the sth component of the discrete Fourier transform. Note that the Fourier transform of $(\nu^2\phi)$ involves ϕ in the real space.

Again splitting the potential and neglecting the backscattered wave, the propagation model for the forward-propagating wave is obtained as

$$2\sqrt{\bar{k}^2-(s\lambda)^2}\phi_{s_x}^+-2i(\bar{k}^2-(s\lambda)^2)\phi_s^+ + [\sqrt{\bar{k}^2-(s\lambda)^2}]_x\phi_s^++i\bar{k}^2F_s(\nu^2\phi^+)=0,$$
$$s=0, \pm1, \pm2, \cdots, \pm(\frac{N}{2}-1), -\frac{N}{2} \quad (23)$$

This equation represents N first-order ordinary differential equations in x, which are solved by a fourth-order Runge-Kutta method. The details of finite differencing and stability analysis of the numerical method are referred to Dalrymple *et al.* (1989). The numerical procedure involves calculating the Fourier modes by marching along the x direction. However, $F_s(\nu^2 \phi^+)$ *in the last term should be calculated in the* real domain, so, at each step, recourse to the real domain by the inverse FFT is needed.

The angular spectrum model (23) is linear since it is based on the linear mild-slope equation (1). In order to incorporate nonlinearity in the model, an empirical nonlinear dispersion relation proposed by Kirby and Dalrymple (1986) is used, which approximates the wavenumber for a solitary wave in shallow water and, in deep water, provides the wavenumber corresponding to the Stokes third-order theory, given by

$$\omega^2 = gk\,(1+f_1\varepsilon^2 D)\tanh(kh+f_2\varepsilon) \tag{24}$$

in which $\varepsilon = ka$; a is the wave amplitude from the linear theory, and

$$f_1(kh) = \tanh^5 kh \tag{25}$$

$$f_2(kh) = \left[\frac{kh}{\sinh kh}\right]^4 \tag{26}$$

$$D = \frac{\cosh 4kh + 8 - 2\tanh^2 kh}{8\sinh^4 kh} \tag{27}$$

The calculation of the wavenumber k using Eq. (24) needs iteration because of the dependence of the wavenumber on wave height. Thus, first the computation is performed with the wavenumber given by the linear dispersion relaion ($f_1 = f_2 = 0$ in Eq. (24)). Using the calculated wave hight, then the wavenumber is corrected by the nonlinear dispersion relation (24). This procedure is repeated until convergence is achieved.

The term $F_s(\nu^2 \phi^+)$ in Eq.(23) represents the interaction between the directional wave modes and the lateral bottom variation, which can force the evolution of the various directional modes, even if they are *initially of zero magnitude. This mechanism can be* explained mathematically by expressing $F_s(\nu^2 \phi^+)$, in terms of periodic convolution (Oppenheim and Schafer, 1975), as

$$F_s(\nu^2\phi^+) = \frac{1}{N}\sum_{p=0}^{N-1} \nu_p^2(x)\,\phi_{s-p}^+(x) \tag{28}$$

in which $\nu_p^2(x)$, p = 0 to (N-1), is the discrete Fourier series obtained by the Fourier transform of $\nu^2(x, y)$ in the y direction. This equation states that the p th bottom mode $\nu_p^2(x)$ triggers the $(s-p)$th wave mode to evolve the s th wave mode. This wave-bottom interaction has been illustrated more explicitly in Dalrymple *et al.* (1989) by applying the model to a monochromatic wave train travelling obliquely over a sinusoidally varying ripple bed.

5. FULLY-NONLINEAR MODEL ON IRREGULAR BATHYMETRY

Starting from the complete boundary value problem for water waves including nonlinear free surface boundary conditions and using the method of multiple scales and Stokes expansions for velocity potential and free surface displacement, Suh (1989) developed an angular spectrum model for propagation of random waves on irregular bathymetry. By doing this, he could include nonlinearity in a more rigorous fashion and express the wave-bottom interaction more exactly than Dalrymple *et al.* (1989). A simple case for a single-frequency wave was reported in Suh, Dalrymple and Kirby (1990). Their model is given by

$$\begin{aligned}
&\frac{\sqrt{k^2-(s\lambda)^2}}{k}C_g A_{s_x} + \frac{[\sqrt{k^2-(s\lambda)^2}\,CC_g]_x}{2\omega}A_s \\
&-i\frac{\omega k\bar{h}}{\sinh 2k\bar{h}}e^{-i\theta_s}I_{s_1} \\
&+i\frac{\omega^2 k^4 \bar{h}^2}{C_g^2\sqrt{k^2-(s\lambda)^2}\sinh^2 2k\bar{h}}E_s e^{-i\theta_s}I_{s_2} \\
&-i\frac{\omega k^2\bar{h}^2}{C_g \sinh 2k\bar{h}}\left[\frac{1}{2}C\tanh k\bar{h} + \frac{g}{\omega\cosh^2 k\bar{h}}\right]e^{-i\theta_s}I_{s_3} \\
&-\frac{g}{2\omega\cosh^2 k\bar{h}}e^{-i\theta_s}I_{s_4} + \frac{1}{g}(G_{33})_s e^{-i\theta_s} = 0
\end{aligned} \tag{29}$$

In this equation, A_s is the complex amplitude of the directional wave mode similar to $\phi_s{}^+$ in Eq.(23), θ_s and Ω_s are the phase functions, I_{s_1} to I_{s_4} are the wave-bottom interaction terms similar to F_s (ν^2 ϕ^+) in Eq. (23), and $(G_{33})_s$ in the last term is the cubic resonant interaction term. More detailed descriptions of these terms are referred to Suh *et al.* (1990). Eq. (29) is N coupled first-order ordinary differential equations for A_s ($s=0, \pm 1, \pm 2,...$) and can be solved by a fourth-order Runge-Kutta method.

Eq. (29) can be compared with Eq.(14) or (23) for simple bathymetry. After substituting for Φ_s^+ by

$$\Phi_s^+ = A_s \exp[i\int^x \sqrt{k^2-(s\lambda)^2}dx] \tag{30}$$

Dalrymple and Kirby's (1988) linear model, (14), becomes

$$\frac{\sqrt{k^2-(s\lambda)^2}}{k}C_g A_{s_x} + \frac{[\sqrt{k^2-(s\lambda)^2}CC_g]_x}{2\omega}A_s = 0 \tag{31}$$

which can be obtained by linearizing Eq.(29) on straight and parallel contours. Thus, Eq. (29) on straight and parallel contours is the nonlinear extension of Dalrymple and Kirby's wide-angle wave propagation model.

The wave-bottom interaction term including F_s (ν^2 ϕ^+) in Dalrymple *et al.* model (23) is replaced by more complicated terms including I_{s_1} to I_{s_4} in Eq. (29). Suh *et al.* (1990) has shown by applying these models to the experiment of Berkhoff *et al.* (1982) that the linearized version of Eq.(29) is computationally much faster and more accurate than the linear form of Dalrymple *et al.* model because of the more refined wave-bottom interaction terms. This is also true when it is compared with a linear parabolic model. The reason why Suh *et al.* model including more complex wave-bottom interaction terms needs less computing time than Dalrymple *et al.* model is that, as can be seen in Eq.(30), the former modeling the slowly varying wave envelopes, A_s can take a larger Δx than the latter modeling the free surface variation, $\Phi_s{}^+$. The computing time of the fully nonlinear equation (29), however, is much greater than that of other nonlinear models owing to the computation of the cubic nonlinear terms $(G_{33})_s$ involving triple summations.

The most advantageous feature of the angular spactrum model is that it permits solution by a marching method like the parabolic model but is valid for waves propagating at large angles from the assumed propagation direction. In order to test the model for waves propagating over an irregular bathymetry at large angles of incidence, Suh *et al.* (1990) have applied the model to the simulation of wave focusing behind a circular shoal resting on a flat bed. Due to the axisymmetry of the circular shoal, the wave focusing pattern behind the shoal should be independent of the angle of incidence, if the model predicts it *correctly*. They concluded that in order for the model to be valid for the case in which waves propagate at large angles from the x direction, the deviation of the actual depth from the laterally-averaged depth should be $O(\epsilon)$ of the laterally-averaged depth.

6. CONCLUDING REMARKS

Recently developed angular spectrum models for water wave propagation have been reported. In all of the models, the wave field at the initial row ($x=0$) is Fourier decomposed into directional modes and the evolution of each mode due to bottom variation is calculated by marching along the x direction; finally, the real wave field is recovered by taking the inverse Fourier transform in the y direction.

The linear version of Suh *et al.* (1990) model is computationally much faster and more accurate than other linear angular spectrum or parabolic models. Its nonlinear version, however, takes more computing time than other nonlinear models. The range of validity of Suh *et al.* model is same as that of third-order Stokes theory. Dalrymple *et al.* (1989) model based on the mild-slope equation, however, can be used in any water depth since the mild-slope equation is exact for deep, shallow and constant-depth water and is valid for intermediate depths as long as the depth does not change too rapidly over a wavelength. Current effects also could be included by using the following dispersion relation

$$\sigma = \omega + \vec{k} \cdot \vec{U}$$

where $\sigma = 2\pi/T$ is newly defined as the angular frequency, ω is given by Eq. (24), and $\vec{U}$ is the depth-mean current vector which is assumed to be known.

The most significant restriction of the angular spectrum model, in the application to a practical problem, due to the use of Fourier transform technique, is that the domain of the model area should be rectangular and should be large enough in the lateral direction to avoid the intrusion of boundary effects into the area of main interest.

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