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Nonparametric Test Procedures for Change Point Problems in Scale Parameter[+]

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ABSTRACT

In this paper we study the properties of nonparametric tests for testing the null hypothesis of no change against one sided and two sided alternatives in scale parameter at unknown point. We first propose two types of nonparametric tests based on linear rank statistics and rank-like statistics, respectively. For these statistics, we drive the asymptotic distributions under the null and contiguous alternatives. The main theoretical tools used for derivation are the stochastic process representation of the test statistic and the Brownian bridge approximation. We evaluate the Pitman efficiencies of the test for the contiguous alternatives, and also compute empirical power by Monte Carlo simulation.

1. Introduction

In the past few years there has been an increasing interest in the problem of detection a possible change in the distribution in a sequence of random observations. For example, they may be a sequence of rates of return per unit time from an investment, or a series of deviations from the target level observed from a quality control process. In these cases, constancy of the variance is important, and the shift of the variance occur at a unknown time point is often something of great practical concern.

Let $X_1, X_2, \cdots, X_N$ denote a successively observed sequence of independent random variables, and let X_i have a continuous cumulative distribution function (cdf) $F_i(x)$, $i=1,\cdots, N$. We assume that there exist a unknown cdf F such that

$$F_i(x)=F\{(x-\mu)/\sigma_i\},\ i=1,\cdots,N, \tag{1}$$

where μ and σ_i are unknown, $-\infty<\mu<\infty$, $\sigma_i>0$.

The null hypothesis of the test is given by

$$H_0: \sigma_1=\cdots=\sigma_N=\sigma_0 \tag{2}$$

and the alternative by

$$H_k: \sigma_1=\cdots=\sigma_k=\sigma_0,\ \ \sigma_{k+1}=\cdots=\sigma_N=\sigma_0+\eta \tag{3}$$

where $k(1\leqq k\leqq N-1)$ is an unknown integer.

Many authors have studied procedures for statistical inferences about change point problems such as detecting a shift in either location or scale parameters and estimation of the change point. The earliest general attempt to treat this problem was made by Page(1954, 1955). He proposed a nonparametric approach for testing hypothesis about single change point in location of distribution using cumulative sums(CUSUMS). Sen and Srivastava(1975) considered several tests using maximum likelihood statistics for detection change in mean with the normality assumption. Hapkins(1977) proposed a likelihood ratio test for the alternative of a location shift and found its distribution under the null hypothesis.

For nonparametric approaches, Bhattacharyya and Johnson(1968) drived the locally best invariant rank test for the distributions which are continuous and symmetric about the shift. Pettitt(1975) proposed a parametric test using a version of the Mann-Whitney statistic. Wolfe and Schechtman(1984) compared various nonparametric tests by Monte Carlo simulation.

A new approach using Brownian bridge process was proposed by MacNeill(1974) and Lombard(1983). MacNeill derived the test statistics using the method of Chernoff and Zack(1984) with the assumption of exponential type distribution and showed that the large sample distribution of the test statistic under th null hypothesis or alternatives is that of a functional on Brownian motion. Lombard extended the MacNeill's results to some class of rank statistics using the weak convergence theorem.

For the change point problem of scale parameter, Hsu(1977, 1979) suggested some procedure in case of normal and gamma distribution and applied these statistics to the analysis of stock market returns and air traffic flows. Talwar and Gentle(1981) proposed simple test based on Pettitt's test which is robust for heavy-tail distribution. Finally, Hsieh(1984) drived a class of rank tests suitable for the scale shift problem without specifying relation between observations and their ordering.

2. Nonparametric Tests for a Change in Scale Parameter

2-1. Test Statistics Based on Linear Rank Statistics

Let $X_1, X_2, \cdots, X_N$ be independent random variables such that X_i have continuous cdf $F_i(x)$ given in (1). And let $R_1, R_2, \cdots, R_N$ denote the corresponding ranks.

We are interested in two sample rank statistics with sample sizes k and N-k defined by

$$S_N(k) = \sum_{i=1}^{k} a_N(R_i) - \frac{k}{N} \sum_{i=1}^{N} a_N(R_i) \tag{4}$$

where $a_N(1), \cdots, a_N(N)$ are scores generated by a function $\phi(u)$, $0 \le u \le 1$,

$$a_N(i) = E(\phi(U_N^{(i)})), \quad 1 \le i \le N \tag{5}$$

with $U_N^{(i)}$ denoting the i-th order statistics in a sample of size N from the uniform distribution on (0, 1).

We consider the null hypothesis H_0 and the alternatives H_k in (2) and (3). Since th integer k is an unknown change point, we must think about a new alternative in this case. This may be the union of the alternatives for the (N−1) two sample problems. Thus we propose two types of test statistics which are suitable for testing the null hypothesis H_0 against new alternative $K = U_{K=1}^{N-1} H_k$.

First, we define the sum-type statistics. Chernoff and Zacks(1964) use these statistic to test null hypothesis in Bayesian setup. Bhattacharyya and Johnson(1968) considered this type of statistics for testing the change point problem with nonparametric approach. These are given by

$$L_N = \sum_{k=1}^{N-1} v_k \cdot (A^2 N)^{-\frac{1}{2}} S_N(k) \tag{6}$$

As a particular case in sum-type statistics, if the weights $v_N(k)$ are $(N-1)^{-1}$ for all k, then statistic L_N is identical to Hsieh's(1984) statistic.

Second, we consider the max-type statistics. Sen and Srivastava(1975) proposed a rank statistic similar to maximum likelihood statistic used in testing change point in parametric model. Here, we can generalize their statistics by

$$M_N = \max_{1 \leq k \leq N-1} w_N(k) \cdot (A^2N)^{\frac{-1}{2}} S_N(k) \qquad (7)$$

where the weight $w_N(k)$'s are nonnegative weight functions.

2-2. Test Statistic Based on Rank-Like Statistics

Most of the nonparametric distribution-free tests are based on rankings of observations in independent random sample from continuous distribution. Thus, under the null hypothesis, such a test uses ranks of independent identically distributed sample observations. However, in the scale problem we consider a new type of test procedures that involve ranking of random variables which, under the appropriate null hypothesis, are exchangeable. We call these statistics rank-like statistics. It was named by Moses (1963) in dealing with the two sample scale problem.

In this section we consider a general approach to the rank-like procedures. Let $X_1, \cdots, X_N$ be indenpendent random variables with continuous cdf $F_i(x)$ given in (1). And let X^* be a symmetric function of the random variables. Here we define new set of random variable $Z_i = | X_i - X^* |$, $i = 1, \cdots, N$. And R_i^*'s are the rank of Z_i's among $Z_1, \cdots Z_N$. Then we define the two types of statistics as follows,

$$L_N^* = \sum_{k=1}^{N-1} v_N^*(k) \cdot (A^2N)^{\frac{-1}{2}} S_N^*(k), \quad M_N^* = \max_{1 \leq k \leq N-1} w_N^*(k) \cdot (A^2N)^{\frac{-1}{2}} S_N^*(k) \qquad (8)$$

where $v_N^*(k)$ and $w_N^*(k)$ are weight functions, and $S_N^*(k)$ is the two sample rank-like statistic based on the ranks $R_1^*, \cdots, R_N^*$.

In particular, if we take weight $w_N^*(k) = 1$ for all k, and score function $a_N(R_i^*)$ being Mann-Whitney-Wilcoxon type, then statistic M_N^* becomes Talwer and Gentle's(1981) statistic.

3. Asymptotic Distributions and Pitman Efficiency

We begin with derviation of large sample distributions for the test statistic under the null hypothesis and a broad range of alternatives applying the theory of weak convergence. The distributions are related to the those of certain functionals on Brownian bridge, but some of the functions appeared in the statistic are not continuous in the uniform topology on [0, 1]. Conditions on functionals are given under which one can assert that the large sample distribution of the test statistic is that of a functional on Brownian bridge. Using these results, we also compute the Pitman efficiencies among the classes of test statistics.

3-1. Sum-Type Statistics

We first consider independent random variables $X_N, \cdots, X_N$ with distribution

$$F_i(x) = F((x-\mu)/e^{\theta_i}), \quad \theta_i = \log \sigma_i, \quad 1 \leq i \leq N.$$

Let us take the null hypothesis as before,

$$H_0 : \theta_1 = \cdots = \theta_N = 0 \qquad (9)$$

and a sequence $\{H_N\}$ of contiguous alternatives to H_0

We define sum-type statistics L_N as in (6) and let

$$\psi(u,\ f)=-1-F^1(u)\cdot\frac{f(F^1(u))}{f(F^1(u))},$$
$$h(s,\ t)=\eta\cdot\{s(1-t)I_{(s\le t)}+t(1-s)I_{(t\le s)}\},\quad 0<s,\ t<1$$
$$\mu_l(t,\ f)=h(s,\ t)\cdot A^{-1}\int_0^1\phi(u)\psi(u,\ f)du, \tag{11}$$

and $[Ns]=max\{k: k\le Ns$ for $s\in[0,\ 1]$ and k is an integer$\}$.

Theorem 3.1 *Let $J(\cdot)$ be a nonnegative measureable function defined on $[0,1]$ such that $\int_0^1(t(1-t))^{1/2}J(t)dt<\infty$. If weight $\{v_N(k)\}$ are given by the integral of function $J(t)$ on interval $I_k=[(2k-1)/(2N),\ (2k+1)/(2N)]$, $k=1,\cdots,\ N-1$, then under $H_{[Ns]}$, the sum-type statistics L_N converges in distribution to $\int_0^1 J(t)(B(t)+\mu_l(t,\ f))dt$, where $B(t)$ is th Brownian bridge.*

(**proof**) We first define the intervals I_k as

$$I_k=[\frac{2k-1}{2N},\ \frac{2k+1}{2N}],\ k=1,\cdots,\ N-1,\ I_0=[0,\ \frac{1}{2N}]\ \text{and}\ I_N=[\frac{2N-1}{2N},\ 1]$$

Then we have

$$S_N([Nt])=S_N(k)\ \text{for}\ t\in I_k,\ k=1,\cdots,\ N-1$$

and the statistic L_N satisfies

$$L_N=\sum_{k=1}^{N-1} v_N(k)\cdot(A^2N)^{\frac{-1}{2}}S_N(k)=\sum_{k=1}^{N-1}\int_{I_k}J(t)\cdot(A^2N)^{\frac{-1}{2}}S_N([Nt])dt$$
$$=\int_{\frac{1}{2N}}^{1-\frac{1}{2N}}J(t)\cdot(A^2N)^{\frac{-1}{2}}S_N([Nt])dt \tag{12}$$

Now we easily show by the weak convergence of linear rank statistic and Lecam's third lemma that, under $H_{[NS]}$, $(A^2N)^{\frac{-1}{2}}S_N([Nt])$ converges in distribution to $B(t)+\mu_l(t,\ f)$.

Since J(t) has a finite integral value on interval $[1/2N,\ 1-1/2N]$ for every N, it follow by the above result that under $H_{[NS]}$, $\int_{\frac{1}{2N}}^{1-\frac{1}{2N}}J(t)\cdot(A^2N)^{-1/2}S_N([Nt])dt$ converge in distribtuion to $\int_{\frac{1}{2N}}^{1-\frac{1}{2N}}J(t)(B(t)+\mu_l(t,\ f))dt$.

On the other hand, since $(\mu_l(t,\ f))^2<t(1-t)$ for $0<s,\ t<1$ and $\int_0^1 J(t)\cdot(t(1-t))^{1/2}dt<\infty$, we have

$$\lim_{N\to\infty} E\left|\int_{I_0\cup I_N}J(t)(B(t)+\mu_l(t,\ f))dt\right|<\lim_{N\to\infty}2\int_{I_0\cup I_N}J(t)(t(1-t))^{\frac{1}{2}}dt=0.$$

Hence we obtain that $\int_{\frac{1}{2N}}^{1-\frac{1}{2N}}J(t)(B(t)+\mu_l(t,f))dt$ coverges in probability to $\int_0^1 J(t)(B(t)+\mu_l(t,\ f))dt$.

Thus if we combine above three results, we obtain that under $H_{[Ns]}$, L_N coverges in distribution to $\int_0^1 J(t)(B(t)+\mu_l(t,\ f))dt$.

From this theorem, the Pitman efficiencies of the sum-type tests follow easily. If we define critical value C_α by

$$P\{\int_0^1 J(t)\cdot B(t)\,dt\le C_\alpha\ \text{for all}\ t\in(0,\ 1)\}=1-\alpha$$

then the asymptotic power of L_N satisfies that

$$\lim_{N\to\infty} P\{L_N>C_\alpha\mid H_{[Ns]}\}=P\{\int_0^1 J(t)(B(t)+\mu_l(t,\ f))dt>C_\alpha\} \tag{13}$$

Theorem 3.2 *Suppose that the weights $\{v_N(k)\}$ satisfy the conditions of theorem (3.1).*

(1) The Pitman efficiency of an L_N-test with score function ϕ_1 and weight generation fucntion $J_1(t)$ with respect to another L_N-test with score function ϕ_2 and weight generation function $J_2(t)$ equals

$$\frac{A^2_{\phi 2}}{A^2_{\phi 1}} \cdot \frac{(\int^1_0 \phi_1(u)\psi(u, f)du)^2}{(\int^1_0 \phi_2(u)\psi(u, f)du)^2} \cdot \frac{(\int^1_0 J_1(t)h(s, t)dt)^2}{(\int^1_0 J_2(t)h(s, t)dt)^2}.$$

(2) The Pitman efficiency of an L_N-test with respect to another L_N-test with the sam score function, but a different weight generating function is

$$\frac{(\int^1_0 J_1(t)h(s, t)dt)^2}{(\int^1_0 J_2(t)h(s, t)dt)^2}$$

3-2. Max-type Statistics

Next, we consider the asymptotic distributions of max-type statistics M_N defined in (7). For these statistics we must distinguish between bounded and unbounded weight functions. Because for the test statistic with bounded weight function it will be easily proved that the max-type statistics converge in distribtion to the maximum of Gaussian processes. But we can not apply similar convergence theorem for the test statistics with unbounded weight. Therefore we have to apply another method for this case.

Now we first consider weight functions $w_N(t)$ such that $0 \le w_N(t) \le c$, *for* $0 \le t \le 1$, some constant c and we define the function $\mathring{w}_N(t) : [0, 1] \to R$ by $\mathring{w}_N(t) = w_N([Nt])$ (*take* $w_N(0) = w_N(N) = 0$).

Theorem 3.3 *Let the function $\mu_1(t, f) : [0, 1] \to R$ be define as in (11). Suppose that the score function $\phi(t)$ is square integrable and the function $\mathring{w}_N$ converges to w in the Skorohod topology. Then under the alternatives $\{H_{[Ns]}\}$, M_N converges in distribution to the supremum of the process $\{w(t)(B(t)+\mu_1(t, f)) : 0 \le t \le 1\}$*

(proof) We define the stochastic process $\{Y_N(t) : t \in [0, 1]\}$ by

$$Y_N(t) = (A^2N)^{\frac{-1}{2}} \cdot S_N([Nt]), \ t \in [0, 1],$$

then we have $M_N = \sup_{0<t<1} \mathring{w}_N(t)\, Y_N(t)$. Following the proof of theorem 3.2 we obtain that $Y_N(t)$ converges in distribution to $B(t) + \mu_1(t, f)$ in the Skorohod topology on D[0, 1].

We define the mapping g and g_N from D[0, 1] to D[0, 1] by

$$g(y) = w(t)y(t), \ g_N(y) = w_N([Nt])y_N(t) \textit{ for every } y \in D[0, 1].$$

Then we have that $M_N = \sup g_N(Y_N(t))$ and by the assumption, $g_N(\cdot)$ coverges to $g(\cdot)$.

Hence it follows that $g_N(Y_N(t))$ converges to $g(B(t) + \mu_1(t, f))$, where $\{B(t) : 0<t<1\}$ is Brownian bridge process on [0, 1].

Now using the result of theorem 3.3 we can derive the Pitman efficiences of max-type tests. If we define the critical value C_α by

$$P\{ \sup_{0<t<1} w(t)B(t) \le C_\alpha \textit{ for all } t \in [0, 1]\} = 1 - \alpha$$

then the asymptotic power of M_N is

$$\lim_{N \to \infty} P\{ M_N > C_\alpha \mid H_{[Ns]}\} = P\{ \sup_{0<t<1} w(t)(B(t) + \mu_1(t, f)) > C_\alpha\} \tag{14}$$

Thus we obtain the following theorem.

(1) The Pitman efficiency of an M_N-test with score function ϕ_1 with respect to another M_N-test with the same weight function, but a different score function ϕ_2 equals to

$$\frac{A^2_{\phi 2}(\int_0^1 \phi_1(u)\psi(u, f)du)^2}{A^2_{\phi 1}(\int_0^1 \phi_2(u)\psi(u, f)du)^2}$$

(2) The Pitman efficiency of an M_N-test with respect to another M_N-test with th same score function, but a different weight function is

$$\frac{sup_{0<t<1}\ w_1(t)}{sup_{0<t<1}\ w_2(t)}$$

Next, we consider the large sample distribution of statistic M_N with unbounded weight function. We are sometimes interested in the weight function

$$w_N(k)=(\frac{k}{N}(1-\frac{k}{N}))^{-1/2}, \quad k=1,\cdots, N-1. \tag{15}$$

For this weight we can not use theorem 3.3. Hence we need to consider another approach. Here we prove that the asymtotic distribution of test statistic with this weight become an extreme value distribution. To do this we need the following definitions : Let

$$a(x)=(2log\ x)^{\frac{1}{2}}, \quad b(x)=2log\ x+2^{-1}\ log_2\ x-2^{-1}\ log\ \pi,$$
$$a_N=(log\ N), \quad b_N=b(iog\ N),$$
$$M_N(x, y)=\max_{k/N\in(x, y)}(\frac{k}{N}(1-\frac{k}{N}))^{\frac{-1}{2}}(A^2N)^{\frac{-1}{2}}\ S_N(k), \tag{16}$$

and

$$M^*_N(x, y)=\max_{u\in(x, y)}(t(1-t))^{\frac{-1}{2}}(A^2N)^{\frac{-1}{2}}\ S_N([Nt]). \tag{17}$$

We further consider the extreme value distribution, $E(x)=\exp(-\exp(-x))$. Then we need a few lemmas in order to prove the main theorem.

Lemma 3.1 *Let $\varepsilon_N>(log\ N)^3/N$. Then there is a sequence of Brownian bridge $\{B(t) : 0\leq t\leq 1\}$ such that*

$$a(log\frac{1-\varepsilon_N}{\varepsilon_N})^{\frac{1}{2}}\cdot\ |M^*_N(\varepsilon_N, 1-\varepsilon_N)-\sup_{\varepsilon_N<t<1-\varepsilon_N}(t(1-t))^{\frac{-1}{2}}B(t)| \tag{18}$$
$$=0((2log_2\ N/log\ N))\ a.s.$$

(proof) From the consequence of Theorem A.2.1 of Cho(1987), there is Brownian bridge $\{B(t) : 0\leq t\leq 1\}$ such that

$$\sup_{0<t<1}|(A^2N)^{\frac{-1}{2}}\ S_N([Nt])-B(t)|=0(log\ N/N^{\frac{1}{2}})\ a.s.$$

Therefore,

$$a_N\cdot\sup_{0<t<1}|(t(1-t))^{\frac{-1}{2}}\ \{(A^2N)^{\frac{-1}{2}}\ S_N([Nt])-B(t)\}|$$
$$=0((2log_2\ N)^{\frac{1}{2}}\cdot log\ N\cdot(log\ N)^{\frac{-3}{2}})\ a.s.$$
$$=0((2log_2\ N/log\ N)^{\frac{1}{2}})\ a.s.$$

Lemma 3.2 *Let $\tau_N=(log\ N)^3/N$. Then we have that $a_N\cdot\max(M^*_N(0, \tau_N), M^*_N(1-\tau_N, 1))-b_N\to-\infty$ in probability.*

$$\lim_{N\to\infty} P\{ \max_{1\leq k\leq(\log N)^3} (\frac{k}{N}(1-\frac{k}{N}))^{\frac{-1}{2}} \cdot (A^2N)^{\frac{-1}{2}} S_N(k)$$
$$> (2\log_2 N)^{\frac{1}{2}} + (2\log_2 N)^{\frac{-1}{2}} \cdot (s+2^{-1}\log_2 N - 2^{-1}\log \pi)\}$$
$$= 0$$

for all $s\in R$. Since

$$\max_{1\leq k\leq(\log N)^3} w_N(k)\cdot(A^2N)^{\frac{-1}{2}} S_N(k) \leq (1-\tau_N)^{\frac{-1}{2}} \max_{1\leq k\leq(\log N)^3} (A^2k)^{\frac{-1}{2}} S_N(k)$$
$$\leq (1+0(1))^{\frac{-1}{2}} \max_{1\leq k\leq(\log N)^3} (A^2k)^{\frac{-1}{2}} S_N(k),$$

it suffices to show that

$$\lim_{N\to\infty} P\{ \max_{1\leq k\leq(\log N)^3} (A^2k)^{\frac{-1}{2}} S_N(k) > (2\log_2 N)^{\frac{1}{2}}\} = 0. \tag{19}$$

For the proof of (19), if we choose some constant $d>1$, $c_0=\log N^3/\log d$ and define $n_j=d^j$. Then we have

$$P\{ \max_{1\leq k\leq N^3} (A^2k)^{\frac{-1}{2}} S_N(k) > (2\log N)^{\frac{1}{2}}\}$$
$$\leq \sum_{j=0}^{[c_0]} P\{ \max_{n_j\leq k\leq n_{j+1}} (A^2k)^{\frac{-1}{2}} S_N(k) > (2\log N)^{\frac{1}{2}}\}$$
$$\leq \sum_{j=0}^{[c_0]} P\{ \max_{n_j\leq k\leq n_{j+1}} (A(N-k))^{-1} S_N(k) > (2n_j \log N)^{\frac{1}{2}}/(N-n_j)\} \tag{20}$$

Here we put $x=(2n_j \log N)^{1/2}/(N-n_j)$. And by using the matingale structures in the scheme of sampling without replacement (Serfling(1974)), we have that for $h>0$,

$$P\{ \max_{n_j< k\leq n_{j+1}} (A(N-k))^{-1} S_N(k) > x\}$$
$$\leq \frac{1}{\exp(hAx)} \cdot E\{\exp(\frac{hS_N(n_{j+1})}{N-n_{j+1}})\}$$
$$\leq \exp\{-xAh + \frac{h^2(1-f^*)n_{j+1}(t-s)^2}{8(N-n_{j+1})^2}\}$$

where $f^*=(n_{j+1}-1)/N$, s and t are any real number. Put $y=(1-f^*)n_{j+1}(s-t)^2/(N-n_{j+1})^2)$. This upper bound is minimal when $h=4xA/y$. Hence we have

$$P\{ \max_{n_j< k\leq n_{j+1}} (A(N-k))^{-1} S_N(k) > x\}$$
$$\leq \exp(\frac{-2(xA)^2}{y}) = N^{-\frac{4A^2}{d(t-s)^2(1-f^*)}} \tag{21}$$

So we have by (20) and (21) that

$$P\{ \max_{1\leq k\leq N^3} (A^2k)^{\frac{-1}{2}} S_N(k) > (2\log N)^{\frac{1}{2}}\} \leq c_0 \cdot N^{-\frac{4A^2}{d(t-s)^2(1-f^*)}} \tag{22}$$

Thus if we take a limit at both sides of the above inequality, we obtain the desired result.

Lemma 3.3 *Let ε_N be a decreasing sequence of number such that ε_N converges to 0. Then we have*

$$\lim_{N\to\infty} P\{a(\log \frac{(1-\varepsilon_N)}{\varepsilon_N}). \sup_{\varepsilon_N<t<1-\varepsilon_N} (t(1-t))^{\frac{-1}{2}} B(t)-b(\log \frac{(1-\varepsilon_N)}{\varepsilon_N})\leq s\}=\mathrm{E}(s) \tag{23}$$

(proof) See Csorge and Pevesz(1981), page 57.

Theorem 3.5 *We assume that the score function* ϕ *is square integrable and the weight function of statistic* M_N *is given in* (15). *Then under* H_0, *for* $-\infty<s<\infty$,

$$\lim_{N\to\infty} P\{a_N\cdot M_N-b_N<s\}=E(s) \tag{24}$$

(Proof) Our proof is in the similar line as in the result of Jaeschke(1979). For $\varepsilon_N>(\log N)^3/N$, we have

$$\sup_{\varepsilon_N<t<1-\varepsilon_N} \frac{Nt(N-Nt)}{[Nt](N-[Nt])}=1+0((\log N)^{\frac{-3}{2}})$$

Thus we have

$$a_N\cdot M_N(\varepsilon_N,\ 1-\varepsilon_N)=a_N\cdot M^{\circ}_N(\varepsilon_N,\ 1-\varepsilon_N)\cdot\{1+0((\log_2 N/(\log N)^3)^{\frac{1}{2}})\}\ a.s.$$

But $a_N\cdot M^{\circ}_N(\varepsilon_N,\ 1-\varepsilon_N)\cdot 0((\log_2 N/(\log N)^3)^{1/2})$ converges to 0 in provability. Thus the theorem 3.5 follows from lemma (3.2) and lemma (3.3).

4. Small Sample Power Comparison of the Tests

4-1. Empirical Critical Values for Test Statistics

In this section, we will estimate the power of test statistics for moderate sample size using Monte Carlo simulation. To do this, we frist have to estimate the empirical critical values of test statistics, and next to estimate the empirical power of tests based on the estimated critical values.

The test statistics we have considered are two kinds of statistics, namely sum-type and max-type based on both rank statistics and rank-like statistics. For each statistic we can take the following score functions. In the rank statistics there are many possible choices for score function $a_N(\cdot)$. Some well known score functions pertinent to testing scale parameter are the Carpon normal score, the Savage exponential score and the Mood score. They are respectively defined by

$$a_N(k)=E(V_N^{(k)})^2,\ a_N(k)=\sum_{j=1}^{k}(N-j+1)^{-1},\ a_N(k)=(\frac{k}{N+1}-\frac{1}{2})^2 \tag{25}$$

where $V^{(k)}_N$ is k-th order statistics from standard normal distribution. In case of rank-like statistics we may take several score functions. Here we only use Wilcoxon score function.

Secondly, we have to take the suitable weight function. In the L_N-test statistics, we can take two possible choices for weight function $v_N(k)$. They are respectively defined by

$$v^1_N(k)=N^{-1} \quad and \quad v^2_N(k)=(k(N-k))^{\frac{-1}{2}},\ k=1,\cdots,\ N-1.$$

Then we define test statistics L_N associated with $v^i_N(k)$ by

$$L^i_N=\sum^{N-1} v^i_N(k)\cdot(A^2N)^{\frac{-1}{2}} S_N(k),\ i=1,\ 2.$$

$$w_N^1(k)=1 \quad \text{and} \quad w_N^2(k)=N\cdot(k(N-k))^{-\frac{1}{2}}, \; k=1,\cdots, N-1.$$

With these weights we define test statistics M_N *as follows*

$$M_N^i=\max_{1<k<N-1} w_N^i(k)\cdot(A^2N)^{-\frac{1}{2}} S_N(k), \; i=1, 2.$$

Finally, we estimate the critical values of the tests for two different sample sizes N=30 and 50 when the nominal levels are α=0.10, 0.05, and 0.01. But we only present the results of sample size 30 in this paper. For each statistic, we replicated 1000 samples of size N. Simulation results are given in Table 1.

Table 1. Empirical Critical Values
sample size=30, No of repetitions=1000

test statistic	weight function	score function	Norminal level		
			0.10	0.05	0.01
sum-type	V_N^1	normal	0.358	0.495	0.810
		Mood	0.362	0.471	0.865
		Savage	0.356	0.472	0.864
		rank-like	0.365	0.476	0.853
	V_N^2	normal	0.836	0.129	1.899
		Mood	0.857	1.134	2.046
		Savage	0.835	1.138	2.039
		rank-like	0.875	1.116	1.976
max-type	W_N^1	normal	0.955	1.125	1.551
		Mood	0.991	1.135	1.723
		Savage	0.976	1.116	1.659
		rank-like	0.974	1.102	1.650
	W_N^2	normal	1.524	1.988	3.793
		Mood	1.571	1.940	4.240
		Savage	2.251	2.604	4.907
		rank-like	1.838	2.822	4.160

4-2. Monte Carlo Simulation

First, we consider four different types of underlying distribution, namely normal, logistic, double-exponential and exponential. In each distribution we have empirically estimated the power for sample size 30 and 50 when the change point occurs between X_k and X_{k+1}, $k=[Nt]$ with $t=0.1$, 0.3 and 0.5. And the scale shift e^{θ} is set to be 1.5, 2.0 and 2.5 and also the significance level is 0.05.

For each alternative, 500 samples of size N were generated and this guarantees that our power estimates are in error by no more than 1.96(0.25/500)=0.043 with 95% confidence. All the simulations were carried out on IBM 4331 computer at Chonnam National University. Here we only present the results of the alternative $e^{\theta}=1.5$ and the chang point at $k=3$ and 15 with sample size 30. These results are shown in the Table 2.

In the sum-type statistics, we see that the normal-score test statistics are most powerful among the considered tests. Moreover, the power of Mood score test is a little better than the power of

Table 2. Monte Carlo Power Comparisons

Sample size 30, $e^0=1.5$

distri-bution	change point	score function	test statistic			
			L_N^1	M_N^1	L_N^2	M_N^2
normal	3	normal	0.113	0.097	0.135	0.145
		Mood	0.088	0.088	0.093	0.061
		Savage	0.093	0.060	0.091	0.069
		rank-like	0.102	0.094	0.111	0.117
	15	normal	0.285	0.287	0.295	0.181
		Mood	0.295	0.295	0.273	0.278
		Savage	0.119	0.079	0.102	0.083
		rank-like	0.280	0.292	0.271	0.275
logistic	3	normal	0.117	0.099	0.147	0.131
		Mood	0.113	0.093	0.113	0.161
		Savage	0.075	0.052	0.079	0.127
		rank-like	0.121	0.095	0.125	0.121
	15	normal	0.223	0.237	0.233	0.116
		Mood	0.267	0.259	0.251	0.253
		Savage	0.081	0.075	0.079	0.079
		rank-like	0.251	0.292	0.245	0.239
double-expon.	3	normal	0.103	0.089	0.123	0.117
		Mood	0.113	0.083	0.111	0.143
		Savage	0.069	0.052	0.075	0.107
		rank-like	0.111	0.111	0.116	0.116
	15	normal	0.176	0.190	0.179	0.095
		Mood	0.204	0.219	0.187	0.201
		Savage	0.075	0.067	0.071	0.073
		rank-like	0.223	0.219	0.219	0.201
exponen.	3	normal	0.063	0.053	0.077	0.060
		Mood	0.083	0.065	0.081	0.091
		Savage	0.107	0.131	0.117	0.151
		rank-like	0.083	0.075	0.093	0.099
	15	normal	0.065	0.073	0.085	0.046
		Mood	0.091	0.093	0.083	0.088
		Savage	0.215	0.190	0.193	0.107
		rank-like	0.165	0.173	0.161	0.147

v^1_N and w^1_N are the most powerful when the change point occurs in the middle part of the sequence, on the other hand the tests with weight v^2_N and w^2_N have high power when the change point is near the ends of two sides in th sequence.

Secondly, we consider the power of L_N and M_N statistics with the random sample coming from a logistic or double-exponential distribution. When the change point occurs in the middle, best power is obtained by test with the Mood score and rank-like statistic with Wilcoxon score for the both weight functions. And the test of normal score has relatively good power. The Savage score test is the least powerful. But when the change point occurs near the beginning point, the high power is obtained by the test with normal score and weights v^2_N and w^2_N. All of other test statistics have similar powers except the Savage score. This result shows that the power of tests depends more on weight functions than the score functions in this case.

Finally, we consider the powers of statistics when distribution is exponential. Th

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