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Sequential Confidence Intervals for the Mean with β−Protection in a Certain Parameter Space+

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ABSTRACT

Let $\{X_n : n=1, 2, \cdots\}$ be iid random variables with distribution P_θ, $\theta \in \Theta$ where Θ is some abstract parameter space. We consider a sequential confidence interval I for the mean $\mu=\mu(\theta)$ of P_θ satisfying $P_\theta(\mu \in I) \geq 1-\alpha$ and $P_\theta(\mu-\delta(\mu) \in I) \leq \beta$ for all $\theta \in \Theta$ for any given an imprecision real valued function $\delta(\mu)>0$ and error probabilities $0<\alpha, \beta<1$.

A one-sided sequential confidence interval is constructed under some restriction on the family $\{P_\theta : \theta \in \Theta\}$ and the imprecision function δ. This is extended to the two-sided case.

1. Introduction

We study sequential procedures for constructing one-sided and bounded sequential confidence intervals for the mean of a distribution in the presence of nuisance parameters.

Sequential experimentation may arise naturally such as in medical trials or may be necessary to achieve the desired precision no matter what the values of the unknown parameters are.

Let $\{X_n : n=1, 2, \cdots\}$ be iid random variables with distribution P_θ, $\theta \in \Theta$. Let the mean $\mu=\mu(\theta)$ of P_θ be the parameter of interest, the rest of θ to be regarded as a nuisance parameter. Any confidence set (CS) to be considered will be subject to two requirements. The first concerns the coverage probability condition i.e. for any given $0<\alpha<1$,

$$P_\theta(\mu \in CS) \geq 1-\alpha \text{ for every } \theta \in \Theta, \tag{1.1}$$

The second requirement concerns the precision of the confidence set, which can be specified in various ways. For example, fixed-width confidence intervals in the univariate case (e.g., Setein(1945) or Chow and Robbins(1965) or minimum risk function(e.g., Wolfowitz(1950)).

In this study, the precision of the confidence sets will be controlled or partly controlled by requiring that with high probability the confidence set does not contain one or more specified parameter values different from the parameter μ of interest. In the simplest case μ is real valued and there is given an imprecision function $\delta(\mu)>0$ and a probability $0<\beta<1$ such that

$$P_\theta(\mu-\delta(\mu) \in CS) \leq \beta \text{ for every } \theta \in \Theta. \tag{1.2}$$

This will be called "β-protection at $\mu-\delta(\mu)$, and was first proposed as a measure of precision by

Wijsman(1981).

A β-protection of the form(1.2) leads to a confidence interval of the form

$$CI=(L(X_1,\cdots, X_N), \infty)$$

Where L is some measurable function of the stopped sequence of random variables when employing a stopping time N. If in (1.2) $\mu-\delta(\mu)$ is replaced by $\mu+\delta(\mu)$, then the appropriate CI has the form $(-\infty, L(X_1,\cdots, X_N))$. Both situations will be called one-sided. In constrast, a two-sided situation arises if (1.2) is replaced by

$$P_\theta(\mu-\delta(\mu)\in CS \text{ or } \mu+\delta(\mu)\in CS)<\beta \text{ for all } \theta\in\Theta.$$

This will be called "β-protection at $\mu\pm\delta(\mu)$. In this case the appropriate CI has the form $(L_1(X_1,\cdots, X_N), L_2(X_1,\cdots, X_N))$.

In the case of no nuisance parameters several studies were done. Wijsman(1982, 1983) treated the mean of a $N(\mu, \sigma^2)$ population with known σ, and of μ/σ for unknown σ with procedures restricted to be scale invariant and Juhlin(1985) studied the mean of a scale parameter exponential distribution. Furthermore Wijsman(1985) studied the one-parameter problem in a rather general setting. This was generalized to vector valued parameters by Fakhre-Zakeri(1987). The purpose of this study is to extend these ideas to problems where nuisance parameters are present.

In this paper we assume that $\{P_\theta: \theta\in\Theta\}$ is the family of all univariate distributions P_θ such that $E_\theta[(X-\mu(\theta))/\sigma(\theta)]^4$ is bounded uniformly in θ, where $X\sim P_\theta$ and $\sigma^2(\theta)=Var_\theta X$.

The imprecision function δ is chosen either constant or δ is strictly monotonic and $\delta(\mu)\to 0$ as $\mu\to-\infty$ plus several smoothness conditions. The case $\delta(\mu)\to 0$ as $\mu\to\infty$, not treated in this paper, is similar. Also the case $\delta(\mu)\to 0$ as $\mu\to\pm\infty$ could be handled by the methods presented here. In all these cases, δ is not bounded away from 0 there is no fixed sample size confidence interval satisfying (1.1) and (1.2) even if μ and σ were the only unknown parameters, so that a truly sequential procedure is mandatory.

Throughout this paper, $\mu(\theta)$ and $\sigma(\mu)$ will abbreviated μ and σ respectively unless specified. We define

$$\bar{X}_n=\frac{1}{n}\sum_{i=1}^{n}X_i \qquad s_n^2=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2 \text{ for } n\geq 2.$$

$$Z_i=(X_i-\mu)/\sigma \quad i=1, 2,\cdots \qquad \bar{Z}_n=\frac{1}{n}\sum_{i=1}^{n}Z_i \text{ so that } \bar{Z}_n=(\bar{X}_n-\mu)/\sigma$$

$$s'^2_n=\frac{1}{n-1}\sum_{i=1}^{n}(Z_i-\bar{Z}_n)^2, \text{ so that } s'^2_n=s^2_n/\sigma^2, \text{ for all } n\geq 2$$

$$\alpha(C, \theta)\equiv P_\theta(\mu\notin I); \quad \beta(C, \theta)\equiv P_\theta(\mu-\delta(\mu)\in I)$$

$$\Theta(\sigma)=\{\theta\in\Theta: \sigma(\theta)=\sigma\}, \quad \Theta(\mu, \sigma)=\{\theta\in\Theta: \mu(\theta)=\mu, \sigma(\theta)=\sigma\}$$

and uniformity in θ will usually be abbreviated "u.i.θ".

2. Sequential Confidence Intervals of Constant Precision

In this section we propose and examine a procedure which satisfies (1.1) and (1.2) with $\delta(\mu)=d$, where $d>0$ is fixed.

Define a stopping rule as

$$N=N(C)=\inf\{n\geq 2: n\geq c^2 s^2_n\} \tag{2.1}$$

$$I=[\bar{X}_N-\rho d,\ \infty) \tag{2.2}$$

in which $0<\rho<1$ is still to be chosen (e.g., $\rho=z_\alpha/(z_\alpha+z_\beta)$).

Lemma 2.1 For arbitary given $\sigma_0>0$, $Nc^{-2}\sigma^{-2}\to 1$ a.s. as $C\to\infty$ u.i. θ if $\sigma(\theta)\geq\sigma_0$.

Proof. By Chow and Robbins (1965, Lemma 1), $N\to\infty$ a.s. as $c\to\infty$ u.i. θ if $\sigma(\theta)\geq\sigma_0$ and by definition of N,

$$s'^2_N\leq Nc^2\sigma^2<s'^2_{N-1}+1/c^2\sigma^2.$$

We can easily show that s'^2_N, $s'^2_{N-1}\to 1$ a.s. as $c\to\infty$ u.i. θ if $\sigma(\theta)\geq\sigma_0$ under the assumption of P_0 and $1/c^2\sigma^2\leq 1/c^2\sigma^2_0\to 0$ as $c\to\infty$ u.i.θ if $\sigma(\theta)\geq\sigma_0$.

Lemma 2.2 $Nc^{-2}\sigma^{-2}\to 1$ a.s. as $\sigma\to\infty$ u.i. $\theta\in\Theta(\sigma)$ for any given $c>0$.

Proof. We can easily show that $N\to\infty$ a.s. as $\sigma\to\infty$ u.i. $\theta\in\Theta(\sigma)$. So the proof is similar to that of Lemma 2.1.

Theorem 2.3 For any given α, β, ρ $(0<\alpha,\ \beta,\ \rho<1)$ there exists $c_0>0$ such that for $c\geq c_0$ the sequential procedure defined by (2.1) and (2.2) satisfies (1.1) and (1.2).

Proof. We will prove the theorem in two parts.

(1) Part 1: There exists $\sigma_0>0$ such that $\alpha(c,\ \theta)\leq\alpha$, $\beta(c,\ \theta)\leq\beta$ u.i.θ if $\sigma(\theta)<\sigma_0$ for all $c>0$.

Proof of Part 1: Observe that

$$\alpha(c,\ \theta)=P_\theta(\bar{Z}_N\geq\rho d/\sigma)\ \textit{and}\ \beta(c,\ \theta)=P_\theta(\bar{Z}_N\leq -d(1-\rho)/\sigma)$$

Let $t=\min(\rho d,\ (1-\rho)d)$ and $\varepsilon=\min(\alpha,\ \beta)$ then

$$\alpha(c,\ \theta)\leq P_\theta(\bar{Z}_N\geq t/\sigma)\ \textit{and}\ \beta(c,\ \theta)\leq P_\theta(\bar{Z}_N\leq -t/\sigma)\ \textit{for all c and}\ \theta.$$

Since $\bar{Z}_n\to 0$ a.s. as $n\to\infty$ u.i. θ. So there is a constant $a>0$ such that $P_\theta\{\bar{Z}_n<a,\ n=1,\ 2,\cdots\}>1-\varepsilon$ for all θ. Hence $P_\theta\{\bar{Z}_N\geq a\}<\varepsilon$ no matter what the stopping time N is. If we choose $\sigma_0=t/a$, then $P_\theta(|\bar{Z}_N|\geq t/\sigma)<\varepsilon$ u.i. θ if $\sigma(\theta)<\sigma_0$.

(2) Part 2: For given $\sigma_0>0$, there exists $c_0>0$ such that $\alpha(c,\ \theta)\leq\alpha$, $\beta(c,\ \theta)\leq\beta$ for all $c\geq c_0$ u.i. θ if $\sigma(\theta)\geq\sigma_0$.

Proof of part 2: We will prove $\alpha(c,\ \theta)$ and $\beta(c,\ \theta)\to 0$ as $c\to\infty$ u.i. θ if $\sigma(\theta)\geq\sigma_0$. Observe that

$$\alpha(c,\ \theta)\leq P_\theta(\sqrt{N}\bar{Z}_N\geq c\rho d s'_N) \tag{2.3}$$

The right-hand side of (2.3) goes to 0 as $c\to\infty$ u.i.θ if $\sigma(\theta)\geq\sigma_0$ by the fact that $\sqrt{N}\bar{Z}_N$ is asymptotically standard normal and therefore stochastically bounded as $c\to\infty$ u.i.θ if $\sigma(\theta)>\sigma_0$ and $c\rho d s'_N\to\infty$ a.s. as $c\to\infty$ u.i.θ.

Similarly $\beta(c,\ \theta)\leq P_\theta(\sqrt{N}\bar{Z}_N\leq -(1-\rho)dcs'_N)\to 0$ as $c\to\infty$ u.i.θ if $\sigma(\theta)\geq\sigma_0$.

If we combine part 1 and part 2 we get c_0 such that $\alpha(c,\ \theta)<\alpha$, $\beta(c,\ \theta)<\beta$ for all $c>c_0$ u.i. θ.

Theorem 2.4 For any fixed $c>0$, as $\sigma\to\infty$ $\alpha(c,\ \theta)\to 1-\Phi(c\rho d)$, $\beta(c,\ \theta)\to 1-\Phi(c(1-\rho)d)$ u.i. $\theta\in\Theta(\sigma)$.

Proof. Write $\alpha(c,\ \theta)$ and $\beta(c,\ \theta)$ as follows

$$\alpha(c,\ \theta)=P_\theta(\sqrt{N}\bar{Z}_N\geq\sqrt{N}\rho d/\sigma)$$

and

$$\beta(c,\ \theta)=P_\theta(\sqrt{N}\bar{Z}_N\leq -\sqrt{N}d(1-\rho)/\sigma)$$

fixed $c>0$ and $\sqrt{N}\rho d/\sigma \xrightarrow{P} c\rho d$, $-\sqrt{N}d(1-\rho)/\rho \xrightarrow{P} -cd(1-\rho)$ as $\sigma\to\infty$. u.i. $\theta\in\Theta(\sigma)$ for any fixed $c>0$. Therefore $\alpha(c,\theta)$ and $\beta(c,\theta)$ converge to $1-\Phi(c\rho d)$ and $\Phi(-c(1-\rho)d)$ respectively.

The extension of Theorem 2.3 to the two-sided case is immediate. Now a confidence interval of the type $[\bar{X}_N-\rho d,\ X_N+\rho d]$ with β-protection at $\mu\pm d$ is needed. This may be achieved by the intersection of $[\bar{X}_N-\rho d,\ \infty)$ obtained according to Theorem 2.3 with α and β replaced by $\alpha/2$, $\beta/2$ respectively and $(-\infty,\ \bar{X}_N+\rho d]$ obtained by the analogue of Theorem 2.3. for such intervals, also with $\alpha/2$, $\beta/2$.

3. Sequential Confidence Intervals of Variable Precision Function

Now we study a sequential procedure with β-protection for a non-constant precision function δ where $\delta: R\to R^+$.

We shall make the following assumption about δ.

Assumption A:

(i) $0<\delta(x)<L$ for all $x\in R$, for some $0<L<\infty$, and $\delta(x)\to 0$ as $x\to-\infty$

(ii) δ is differentiable and $0<\delta'(x)<M$ for all $x\in R$ for some $0<M<\infty$

(iii) $\delta(x+y)/\delta(x)\to 1$ as $y\to 0$ uniformly in x

(iv) for any given $\varepsilon>0$, $B>0$, $0<\xi\leq 1$, there exists x_0 such that

$$(1+\varepsilon)\ \delta(x)-\delta(x+B/(\delta(x))^{\xi})>0 \text{ for all } x\leq x_0.$$

Lemma 3.1 For any given $a>b>0$, $0<\xi\leq 1$ and $B>0$ there exists x^* such that $[\delta(x+B/(\delta(x))^{\xi})-\delta(x)]^2/[a\delta(x)-b\delta(x+B/(\delta(x))^{\xi})]^2<1/b^2$ for all $x\leq x^*$.

Proof: Consider the identity

$$b^2[\delta(x+B/(\delta(x))^{\xi})-\delta(x)]^2-[a\delta(x)-b\delta(x+B/(\delta(x))^{\xi})]^2$$
$$=-(a-b)\ \delta(x)\ [(a+b)\delta(x)-2b\delta(x+B/(\delta(x))^{\xi})]$$

valid for all x and ξ. Now in Assumption A(iv) put $\varepsilon=(a+b)/2b-1$ and choose x_0 according to A(iv). With $x^*=x_0$ we have then that $(a+b)\ \delta(x)-2b\delta(x+B/(\delta(x))^{\xi})>0$ for all $x<x^*$.

A reasonable stopping time N may be proposed as

$$N=N(c)=\inf\{n\geq 2:\ n\geq c^2s_n^2/\delta^2(\bar{X}_n)\},\ c>0 \tag{3.1}$$

and our terminal decision rule is the confidence interval

$$I=[\bar{X}_N-\rho\delta(\bar{X}_N),\ \infty) \tag{3.2}$$

with $\rho(0<\rho<1)$ to be chosen.

Throughout this Section, we partition the parameter space into three disjoint regions. For arbitrary given $d>0$ and $\sigma^*>0$

$$A_1=\{\theta\in\Theta:\ 0<\sigma(\theta)<d\delta(\mu(\theta))\}$$
$$A_2=\{\theta\in\Theta:\ \sigma^*\geq d\delta(\mu(\theta))\}$$
$$A_3=\{\theta\in\Theta:\ \sigma(\theta)\geq\sigma^*\} \text{ and define } N_0=c^2\sigma^2/\delta^2(\mu)$$

with $[x]$ will be meant the greatest integer not exceeding x.

Thoerem 3.2 If δ satisfies Assumption A, then for any given $\alpha,\beta,\rho(0<\alpha,\beta,\rho<1)$ there exists $c_0>0$ such that with the stopping time (3.1) and confidence interval (3.2) for the mean μ, (1.1) and (1.2) are achieved for all $c\geq c_0$.

Before proving the theorem, we need the following lemmas.

Lemma 3.3 $\delta^2(\mu)\ s'^2_N/\delta^2(\mu+\sigma\bar{Z}_N)\to 1$ a.s. as $c\to\infty$ u.i. $\theta\in A_2$.

$\theta \in A_2$.

Therefore $\sigma\bar{Z}_N \to 0$, $s^2_N \to 1$ a.s. as $c \to \infty$ u.i. $\theta \in A_2$. The Lemma holds, by Assumption A(iii).

Lemma 3.4 Lemma 3.3 is valid with N replaced by N−1.

Proof: The proof of Lemma 3.3 is unchanged when replacing N by N−1 since $s^2_{N-1} \to 1$, $\sigma\bar{Z}_{N-1} \to 0$ a.s. as $c \to \infty$ u.i. $\theta \in A_2$.

Lemma 3.5 $N/N_0 \to 1$ a.s. as $c \to \infty$ u.i. $\theta \in A_2$.

Proof: By definition of N the following double inequality on N/N_0 holds.

$$\delta^2(\mu)s'^2_N/\delta^2(\mu+\mu\bar{Z}_N) \leq N/N_0 < \delta^2(\mu)s'^2_{N-1}/\delta^2(\mu+\sigma\bar{Z}_{N-1}) + \delta^2(\mu)/c^2\sigma^2.$$

By Lemmas 3.3, 3.4 and $\delta^2(\mu)/c^2\sigma^2 \leq 1/c^2d^2 \to 0$ as $c \to \infty$ u.i. $\theta \in A_2$, the Lemma hold.

Lemma 3.6 $N \to \infty$ a.s. as $c \to \infty$ u.i. $\theta \in A_3$.

Proof: Define a new stopping time N_1 as follows;

$$N_1 = N_1(c, \theta) = \inf\{n > 2 :\ n > c^2\sigma^2 s'^2_n/L^2\}.$$

Then $N_1 < N$ since δ is bounded by L. Since $N_1 \to \infty$ a.s. as $c \to \infty$ u.i. $\theta \in A_3$, so does N.

Lemma 3.7. For arbitrary given ε $(0<\varepsilon<1)$ $P_\theta\{N > (1+\varepsilon)N_0\} \to 0$ as $c \to \infty$ u.i. $\theta \in A_3$.

Proof: By definition of N in (3.1)

$$\begin{aligned} P_\theta(N > (1+\varepsilon)N_0) &= P_\theta\{\sqrt{n}\,\delta(\bar{X}_n) < cs'_n \text{ for all } n \leq (1+\varepsilon)N_0\} \\ &\leq P_\theta\{\sqrt{n_0}(\delta(\mu+\sigma\bar{Z}_{n_0}) - \delta(\mu)) \leq c\sigma s'_{n_0} - \sqrt{n_0}\ \delta(\mu)\} \end{aligned} \tag{3.3}$$

where $n_0 = [(1+\varepsilon)N_0]$.

Write $\delta(\mu+\sigma\bar{Z}_{n_0}) - \delta(\mu) = \delta'(V_{n_0})\sigma\bar{Z}_{n_0}$ where $|V_{n_0} - \mu| < |\sigma\bar{Z}_{n_0}|$. Observe that $n_0 > (1+\varepsilon)N_0 - 1 = c^2\sigma^2/\delta^2(\mu)\ [1+\varepsilon-\delta^2(\mu)/c^2\sigma^2]$. Hence the right-hand side of (3.3) can be written as

$$\begin{aligned} &P_\theta\{\sqrt{n_0}\delta'(V_{n_0})\sigma\bar{Z}_{n_0} < c\sigma s'_{n_0} - \sqrt{n_0}\delta(\mu)\} \leq \\ &P_\theta\{\sqrt{n_0}\delta'(V_{n_0})\bar{Z}_{n_0} < c[1+\varepsilon/3 - (1+\varepsilon-\delta^2(\mu)/c^2\sigma^2)^{1/2}]\} \\ &+ P_\theta\{s'_{n_0} > 1+\varepsilon/3\}. \end{aligned}$$

Observe that $n_0 \to \infty$ as $c \to \infty$ u.i. $\theta \in A_3$. Therefore $s'_{n_0} \to 1$ a.s. and $\sqrt{n_0}\bar{Z}_{n_0} \to N(0, 1)$ as $c \to \infty$ u.i. $\theta \in A_3$. Define $\varepsilon_1(c, \theta) = 1+\varepsilon/3-(1+\varepsilon-\delta^2(\mu)/c^2\sigma^2)^{1/2}$. *Then* $\varepsilon_1(c, \theta) < 0$ for sufficiently large c uniformly in $\theta \in A_3$ and also $c\varepsilon_1 \to -\infty$ as $c \to \infty$ u.i. $\theta \in A_3$. Since δ' is bounded above by M, $c\varepsilon_1/\delta'(V_{n_0}) \to -\infty$ as $c \to \infty$ u.i. $\theta \in A_3$. Therefore $P_\theta\{s'_{n_0} > 1+\varepsilon/3\} \to 0$ as $c \to \infty$ u.i. $\theta \in A_3$.

Lemma 3.8 For arbitrary given $\varepsilon(0<\varepsilon<1)$, μ_0, $\sigma^* > 0$, as $c \to \infty$ $P_\theta\{N < (1-\varepsilon)N_0\} \to 0$ u.i. $\theta \in \{\theta \in \Theta :\ \mu(\theta) > \mu_0,\ \sigma(\theta) > \sigma^*\}$.

Proof:

$$\begin{aligned} &P_\theta(N \leq (1-\varepsilon)N_0) \\ &\leq P_\theta\{n\delta^2(\bar{X}_n) \geq c^2 s^2_n \text{ for some } c^2\sigma^2/L^2 \leq n \leq (1-\varepsilon)N_0\} \\ &\quad + P_\theta\{s'^2_N < 1-\varepsilon\} \\ &\leq P_\theta\{n\delta^2(\bar{X}_n) \geq c^2\sigma^2(1-\varepsilon/2),\ \text{for some } n \leq (1-\varepsilon)N_0\} \\ &\quad + P\{s'^2_n < 1-\varepsilon/2,\ \text{for some } n \geq c^2\sigma^2/L^2\} + P_\theta\{s'^2_N < 1-\varepsilon\}. \end{aligned}$$

Put $A_4 = \{\theta \in \Theta :\ \mu(\theta) \geq \mu_0,\ \sigma(\theta) \geq \sigma^*\}$.

0 as $c\to\infty$ u.i. $\theta\in A_3$ and therefore in A_4.

On the other hand

$$P_\theta\{n\delta^2(\bar{X}_n)\geq c^2\sigma^2(1-\varepsilon/2),\ \textit{for some}\ n\leq(1-\varepsilon)N_0\}$$
$$\leq P_\theta\{n(\delta^2(\bar{X}_n)-\delta^2(\mu))>c^2\sigma^2(1-\varepsilon/2)-n\delta^2(\mu),$$
$$\textit{for some}\ n<(1-\varepsilon)N_0\}. \tag{3.4}$$

Write $\delta^2(\bar{X}_n)-\delta^2(\mu)=2\delta(V_n)\delta'(V_n)\sigma\bar{Z}_n$, where $|V_n-\mu|<|\sigma\bar{Z}_n|$.

Thus the right-hand side of (3.4) is equivalent to

$$P_\theta\{2n\delta(V_n)\delta'(V_n)\sigma\bar{Z}_n\geq c^2\sigma^2(1-\varepsilon/2)-n\delta^2(\mu),$$
$$\textit{for some}\ n\leq(1-\varepsilon)N_0\}$$
$$\leq P_\theta\{\max_{n\leq(1-\varepsilon)N_0}|n\bar{Z}_n|\geq c^2\sigma\varepsilon/(4LM)\}\leq 16L^2M^2(1-\varepsilon)/$$
$$(\varepsilon^2c^2\delta^2(\mu))\to 0\ \textit{as}\ c\to\infty\ \textit{u.i.}\ \theta\in A_4.$$

Corollary 3.9 For arbitrary given μ_0, $\sigma^*>0$, $N/N_0\to 1$ as $c\to\infty$ u.i. $\theta\in\{\theta\in\Theta:\ \mu(\theta)\geq\mu_0,\ \sigma(\theta)>\sigma^*\}$.

Proof: The corollary follows immediately from Lemmas 3.7, 3.8.

Lemma 3.10 Let $0<\varepsilon<1$, $\sigma^*>0$ and $0<\xi\leq1$ be given. Define $N_\infty=c^2\sigma^2/(\delta(\mu))^{2\xi}$. Then there exists μ^* such that $P_\theta\{N\leq(1-\varepsilon)N_\infty\}\to 0$ as $c\to\infty$ u.i. $\theta\in A_5\equiv\{\theta\in\Theta:\ \mu(\theta)<\mu^*,\ \sigma(\theta)\geq\sigma^*\}$.

Proof: Write $\delta(\mu+\sigma\bar{Z}_n)-\delta(\mu)=\delta'(V_n)\sigma\bar{Z}_n$, where $|V_n-\mu|<\sigma\bar{Z}_n$. Then

$$P_\theta(N\leq(1-\varepsilon)N_\infty)$$
$$\leq P_\theta\{n\delta^2(\bar{X}_n)\geq c^2\sigma^2(1-\varepsilon/2),\ \textit{for some}\ n\leq(1-\varepsilon)N_\infty\}$$
$$+P_\theta\{s'^2_n<1-\varepsilon/2,\ \textit{for some}\ n>c^2\sigma^2/L\}+P_\theta\{s'^2_N<1-\varepsilon\}. \tag{3.5}$$

As in the proof of Lemma 3.8,
$P_\theta\{s'^2_n<1-\varepsilon/2$, for some $n\geq c^2\sigma^2/L^2\}+P_\theta\{s'^2_N<1-\varepsilon\}\to 0$ as $c\to\infty$ u.i. $\theta\in A_5$. Next we must prove that the first term of right-hand of (3.5) goes to zero as $c\to\infty$ u.i. $\theta\in A_5$ where μ^* will be determined later. Write the first term of right-hand side of (3.5) as

$$P_\theta\{n(\delta^2(\mu+\sigma\bar{Z}_n)-\delta^2(\mu))\geq c^2\sigma^2(1-\varepsilon/2)-n\delta^2(\mu),$$
$$\textit{for some}\ n\leq(1-\varepsilon)N_\infty\}. \tag{3.6}$$

Observe $c^2\sigma^2(1-\varepsilon/2)-n\delta^2(\mu)\geq c^2\sigma^2\varepsilon/2$ for all $n\leq(1-\varepsilon)N_\infty$ and $\mu\leq\mu_0$, where $\mu_0=\sup\{\mu:\delta(\mu)\leq1\}$. For $\{\theta\in\Theta:\ \mu(\theta)\leq\mu_0\}$ (3.6) is less than

$$P_\theta\{n(\delta(\mu+\sigma\bar{Z}_n)-\delta(\mu))\ (\delta(\mu+\sigma\bar{Z}_n)+\delta(\mu))>c^2\sigma^2\varepsilon/2,$$
$$\textit{for some}\ n\leq(1-\varepsilon)N_\infty\}. \tag{3.7}$$

By the monotonicity of δ (Assumption A(ii)), (3.7) is equal to

$$P_\theta\{n(\delta(\mu+\sigma\bar{Z}_n)-\delta(\mu))\ (\delta(\mu+\sigma\bar{Z}_n)+\delta(\mu))$$
$$\geq c^2\sigma^2\varepsilon/2,\ Z_n\geq0,\ \delta'(V_n)\leq(\delta(\mu))^\xi,\ \textit{for some}\ n\leq$$
$$(1-\varepsilon)N_\infty\}+P_\theta\{n(\delta(\mu+\sigma\bar{Z}_n)-\delta(\mu))\ (\delta(\mu+\sigma\bar{Z}_n)$$
$$+\delta(\mu))\geq c^2\sigma^2\varepsilon/2,\ \bar{Z}_n\geq0,\ \delta'(V_n)\geq(\delta(\mu))^\xi,$$
$$\textit{for some}\ n\leq(1-\varepsilon)N_\infty\}. \tag{3.8}$$

The first term of (3.8) is less than

$$P_\theta\{n\bar{Z}_n\geq c^2\sigma\varepsilon/((4(\delta(\mu))^\xi L),\ \textit{for some}\ n\leq(1-\varepsilon)N_\infty\}$$

since $\delta(\mu+\sigma\bar{Z}_n)-\delta(\mu)=\delta'(V_n)\sigma\bar{Z}_n\leq(\delta(\mu))^{\xi}\sigma\bar{Z}_n$ and $\delta(\mu+\sigma\bar{Z}_n)+\delta(\mu)<2L$.

In the second term of (3.8) $\delta'(V_n)>(\delta(\mu))^{\xi}$ and $\bar{Z}_n\geq0$ imply $0<\sigma\bar{Z}_n<L/(\delta(\mu))^{\xi}$ since $L>\delta(\mu+\sigma\bar{Z}_n)-\delta(\mu)=\delta'(V_n)\ \sigma\bar{Z}_n$ and also $\delta(\mu+\sigma\bar{Z}_n)+\delta(\mu)=2\delta(\mu)+\delta'(V_n)\sigma\bar{Z}_n<2\delta(\mu)+M\sigma\bar{Z}_n$ since $\delta'<M$. $\leq M$.

So the second term of (3.8) is less than

$$P_\theta\{n(\delta(\mu+L/(\delta(\mu))^{\xi})-\delta(\mu))\ (2\delta(\mu)+M\sigma\bar{Z}_n)$$
$$\geq c^2\sigma^2\varepsilon/2,\ \textit{for some}\ n\leq(1-\varepsilon)N_\infty\}$$
$$=P_\theta\{\sigma n\bar{Z}_n>K/M,\ \textit{for some}\ n<(1-\varepsilon)N_\infty\} \tag{3.9}$$

where $K=c^2\sigma^2\varepsilon/2[\delta(\mu+L/(\delta(\mu))^{\xi})-\delta(\mu)]-2n\delta(\mu)$.

But for $\{\theta\in\Theta:\ \mu(\theta)\leq\mu_0\}$ and $n<(1-\varepsilon)N_\infty$

$$K\geq c^2\sigma^2\{\varepsilon/2[\delta(\mu+L/(\delta(\mu))^{\xi})-\delta(\mu)]-2(1-\varepsilon)/\delta(\mu)\}$$
$$=\frac{c^2\sigma^2}{2\delta(\mu)}\frac{(4-3\varepsilon)\ \delta(\mu)-4(1-\varepsilon)\ \delta(\mu+L/(\delta(\mu))^{\xi})}{\delta(\mu+L/(\delta(\mu))^{\xi})-\delta(\mu)}\equiv c^2\sigma^2/2\delta(\mu)R,\ \textit{say}.$$

Then by Lemma 3.1 and Assumption A(iv), there exists μ_{00} such that $R\geq4(1-\varepsilon)$ for all $\mu<\mu_{00}$. So define $\mu^*=\min\{\mu_0,\ \mu_{00}\}$. Then for all $\{\theta\in\Theta:\ \mu(\theta)<\mu^*\}$ (3.9) is less than

$$P_\theta\{\max_{n<(1-\varepsilon)N_{00}}|n\bar{Z}_n|>2c^2\sigma(1-\varepsilon)/M\delta(\mu)\}$$
$$\leq[M\delta(\mu)/(2c^2\sigma(1-\varepsilon))]^2\ (1-\varepsilon)N_\infty$$
$$\leq M^2/(4(1-\varepsilon)c^2)\rightarrow0\ \textit{as}\ c\rightarrow\infty.$$

Lemma 3.11 For arbitrary given $\sigma^*>0$ there exists μ^* such that $\sigma\bar{Z}_n\rightarrow0$ a.s. as $c\rightarrow\infty$ u.i. $\theta\in A_5$, where A_5 is defined in Lemma 3.10.

Proof: Let μ^*, N_∞ be the same as defined in Lemma 3.10. Define $\gamma=\min\{k:$ positive integer, $2\xi\geq(1/2)^k\}$. Then for arbitrary given $\varepsilon>0$.

$$P_\theta\{|\sigma\bar{Z}_N|>\varepsilon\}\leq P_\theta\{N\leq(1-\varepsilon)N_\infty\}+P_\theta\{N>(1+\varepsilon)N_0\}$$
$$+\sum_{A(1)<j\leq(1+\varepsilon)N_0}P_\theta\{N=j,\ |\sigma\bar{Z}_j|>\varepsilon\}$$
$$+\sum_{m=1}^{\gamma}(\sum_{A(m+1)<j\leq A(m)}P_\theta\{N=j,\ |\sigma\bar{Z}_j|>\varepsilon\})$$
$$+\sum_{(1-\varepsilon)N_{00}<j\leq A(\gamma+1)}P_\theta\{N=j,\ |\sigma\bar{Z}_j|>\varepsilon\} \tag{3.10}$$

where $A(m)=c^2\sigma^2/(\delta(\mu))^{(1/2)^{m-1}}$

For any given $0<a<b$,

$$\sum_{a<j<b}P_\theta(N=j,\ |\sigma\bar{Z}_j|>\varepsilon)\leq P_\theta(\max_{a<j<b}|j\bar{Z}_j|>a\varepsilon/\sigma)<(\sigma/a\varepsilon)^2b.$$

Thus the third term on the right-hand side of (3.10) is less than $(1+\varepsilon)/c^2\varepsilon^2$. Similarly the fourth and fifth term on the right-hand side of (3.10) are less than $\gamma/c^2\varepsilon^2$, $(\delta(\mu))^{2\xi}/c^2\varepsilon^2(1-\varepsilon)^2$ respectively. Thus

$$P_\theta(|\sigma\bar{Z}_N|>\varepsilon)\leq P_\theta(N\leq(1-\varepsilon)N_\infty)+P_\theta(N>(1+\varepsilon)N_0)+(1+\varepsilon+\gamma)/c^2\varepsilon^2$$
$$+(\delta(\mu))^{2\xi}/c^2\varepsilon^2(1-\varepsilon)^2\rightarrow0\ \textit{as}\ c\rightarrow\infty\ \textit{u.i.}\ \theta\in A_5$$

since $P_\theta(N\leq(1-\varepsilon)N_\infty)+P_\theta(N>(1+\varepsilon)N_0)\rightarrow0$ as $c\rightarrow\infty$ u.i. $\theta\in A_5$ by Lemmas 3.8 and 3.10.

Lemma 3.12 Lemma 3.11 is valid after replacing N by N−1.

$$\sum_{a<j<b} P_\theta(N=j,\ |\bar{Z}_{j-1}|>\varepsilon)<P_\theta(\max_{a<j<b} |(j-1)\bar{Z}_{j-1}|>\varepsilon(a-1)/\sigma)$$
$$\leq(\sigma/\varepsilon(a-1))^2 b=(\sigma/a\varepsilon)^2 b\ )/(1-1/a)^2.$$

So for any given $\varepsilon>0$

$$P_\theta(|\sigma\bar{Z}_{N-1}|>\varepsilon)<P_\theta(N\leq(1-\varepsilon)N_{00})+P_\theta(N\geq(1+\varepsilon)N_0)+[(1+\varepsilon+\gamma)/c^2\varepsilon^2$$
$$+(\delta(\mu))^{2\xi}/c^2\varepsilon^2(1-\varepsilon)^2]/(1-(\delta(\mu))^{2\xi}/c^2\sigma^2)^2\to 0 \text{ as } c\to\infty \text{ u.i.}$$
$$\theta\in A_5 \text{ since } 1-(\delta(\mu))^{2\xi}/c^2\sigma^2\to 1 \text{ as } c\to\infty \text{ u.i. } \theta\in A_5.$$

Corollary 3.13 For arbitrary given $\delta^*>0$, there exists μ^* such that $N/N_0\to 1$ in probability as $c\to\infty$ u.i. $\theta\in A_5$, where A_5 is defined in Lemmma 3.10.

Proof: s'^2_N, $s'^2_{N-1}\to 1$ a.s. as $c\to\infty$ u.i. $\theta\in A_3$ and therefore in A_5 since A_5 is a subset of A_3. Also by Lemmas 3.11, 3.12 and Assumption A(iii), $\delta^2(\mu)/\delta^2(\mu+\sigma\bar{Z}_n)$ and $\delta^2(\mu)/\delta^2(\mu+\delta\bar{Z}_{N-1})$ converge to 1 a.s. as $c\to\infty$ u.i. $\theta\in A_5$.

Proof of Theorem 3.2

First we will show that there exists $d>0$ such that $\alpha(c,\theta)<\alpha$, $\beta(c,\theta)<\beta$ for all θ if $\delta(\theta)<d\delta(\mu(\theta))$, $c>0$. Using confidence interval (3.2),

$$\begin{aligned}\alpha(c,\theta)&=P_\theta(\bar{Z}_N>\rho\delta(\mu+\sigma\bar{Z}_N)/\sigma)\\&=P_\theta(\bar{Z}_N>\rho\delta(\mu+\sigma\bar{Z}_N)/\sigma,\ \bar{Z}_N>0)\\&<P_\theta(\bar{Z}_N>\rho\delta(\mu)/\sigma) \text{ by Assumption } A(ii)\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}\beta(c,\theta)&=P_\theta(\bar{Z}_N<(\rho\delta(\mu+\sigma\bar{Z}_N)-\delta(\mu))/\sigma)\\&=P_\theta(\bar{Z}_N<(\rho\delta(\mu+\sigma\bar{Z}_N)-\delta(\mu))/\sigma,\ \bar{Z}_N>0)\\&+P_\theta(\bar{Z}_N<(\rho\delta(\mu+\sigma\bar{Z}_N)-\delta(\mu))/\sigma,\ \bar{Z}_N<0)\end{aligned}\tag{3.12}$$

If $Z_N\geq 0$, write $\delta(\mu+\sigma\bar{Z}_N)=\delta(\mu)+\delta'(V_N)\sigma\bar{Z}_N\leq\delta(\mu)+M\sigma\bar{Z}_N$, where $|V_N-\mu|<\sigma\bar{Z}_N$, if $\bar{Z}_N<0$, $\delta(\mu+\sigma Z_N)<\delta(\mu)$ by the monotonicity of δ.

Therefore (3.12) can be bounded above either by

$$P_\theta(\bar{Z}_N>(1-\rho)\delta(\mu)/(\rho M-1)\sigma)+P_\theta(\bar{Z}_N<-(1-\rho)\delta(\mu)/\sigma)$$
$$\text{if } \rho M-1>0$$

or by

$$P_\theta(\bar{Z}_N<(1-\rho)\delta(\mu)/\sigma) \text{ if } \rho M-1\leq 0.$$

$$\text{Define } t=\begin{cases}\min(\rho,\ 1-\rho) & \text{if } \rho M-1\leq 0\\ \min(\rho,\ (1-\rho),\ (1-\rho)/(\rho M-1)\ ) & \text{if } \rho M-1>0\end{cases}$$

Then for all $c>0$, θ

$$\alpha(c,\theta)\leq P_\theta(\bar{Z}_N>t\delta(\mu)/\sigma)\tag{3.13}$$

and

$$\beta(c,\theta)\leq P_\theta(|\bar{Z}_N|>t\delta(\mu)/\sigma)\tag{3.14}$$

Define $\varepsilon=\min(\alpha,\beta)$. Since $\bar{Z}_N\to 0$ a.s. as $n\to\infty$ u.i. θ there exists a constant $a>0$ such that $P_\theta(|\bar{Z}_N|>a)<\varepsilon$ for all $\theta\in\Theta$ no matter what the stopping time N is. If we choose $a=t\,\sigma(\mu)/\sigma$, then (3.13) and (3.14) are less than ε for all θ if $\sigma(\theta)<t/a\delta(\mu)$. So we take $d=t/a$

$$P_\theta(\mid \bar{Z}_N \mid > t\delta(\mu)/\sigma) = P_\theta(\mid \sqrt{N}\bar{Z}_N \mid > t\sqrt{N}\delta(\mu)/\sigma) \tag{3.15}$$

Since $\sqrt{N}\,\bar{Z}_N$ is stochastically bounded as $c\to\infty$ u.i. θ if $\sigma(\theta)>d\delta(\mu(\theta))$ and $t\sqrt{N}\,\delta(\mu)/\sigma = t(N/N_0)^{1/2}\, N_0^{1/2}\,\delta(\mu)/\sigma = ct(N/N_0)^{1/2}\to\infty$ a.s. as $c\to\infty$ u.i. θ if $\sigma(\theta)>d\delta(\mu(\theta))$ by Corollaries 3.9, 3.13 and Lemma 3.5. Thus (3.15) goes to 0 as $c\to\infty$ u.i. θ if $\sigma(\delta)>d\delta(\mu(\theta))$.

The extension of Theorem 3.2 to the two-sided case is immediate as in Section 2.

Theorem 3.14 *For any given $c>0$, and $\sigma>0$, as $\mu\to-\infty$, $\alpha(c,\theta)\to 1-\Phi(\rho c)$, $\beta(c,\theta)\to 1-\Phi(c(1-\rho))$ u.i. $\theta\in\Theta(\mu,\sigma)$, where the sequential procedure is defined by (3.1) and (3.2).*

Proof: $\alpha(c,\theta) = P_\theta(\sqrt{N}\bar{Z}_N > \sqrt{N}\ \rho\delta(\mu+\sigma\bar{Z}_N)/\sigma)$

and

$$\beta(c,\theta) = P_\theta(\sqrt{N}\bar{Z}_N < \sqrt{N}(\rho\delta(\mu+\sigma Z_N) - \delta(\mu))/\sigma)$$

We can easily show that $\sqrt{N}\bar{Z}_N \to N(0, 1)$ as $\mu\to-\infty$ u.i. $\theta\in\Theta(\mu,\sigma)$ for any given $c>0$, $\sigma>0$ and also $\sqrt{N}\ \rho\delta(\mu+\sigma\bar{Z}_N)/\sigma = (N/N_0)^{1/2}\ \rho c d(\mu+\sigma\bar{Z}_N)/\delta(\mu)\to\rho c$ a.s. as $\mu\to-\infty$ u.i. $\theta\in\Theta(\mu,\sigma)$ for any given c. $\sqrt{N}(\rho\delta(\mu+\sigma\bar{Z}_N)-\delta(\mu))/\sigma\to -c(1-\rho)$ a.s. as $\mu\to-\infty$ u.i. $\theta\in\Theta(\mu,\sigma)$ for any given $c>0$, $\sigma>0$.

Remark 3.15 For any given μ and $c>0$, as $\sigma\to\infty$ $\alpha(c,\theta)$ and $b(c,\theta)$ may not have limiting values for the sequential procedure defined by (3.1) and (3.2). Furthermore N/N_0 does not converge in probability to a constant.

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