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Note on Truncated Estimators in Recovery of Interblock Information

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ABSTRACT

In the problem of the recovery of interblock information in balanced incomplete block designs(BIBD), it is well known that a truncated estimator dominates the untruncated estimator under mean squared error loss. This paper shows that the domination result holds universally for every nondecreasing loss function.

1. Introduction

Consider a balanced incomplete block design with both blocks and errors random. Let t= number of treatments, b=number of blocks, r=number of replications, k=number of cells per block, and λ=number of times any pair of treatments appears in the same block. Throught the paper, we use the following notations:

$$\alpha=k/(\lambda t),\ \beta=k/(r-\lambda),\ \gamma=1/(bk),$$
$$p=t-1,\ m=bk-b-t+1,\ n=b-t,$$
$$\theta=\alpha\sigma_1^2/(\alpha\sigma_1^2+\beta\alpha_2^2),\ \theta_0=\alpha/(\alpha+\beta).$$

These constants satisfy $\alpha>0, \beta>0, \gamma>0, p\geq 2, m\geq 1, n\geq 1$. Then Graybill and Weeks(1959) give the following canonical form: The statistics $X_1,\cdots,X_p$, Y_0, $Y_1,\cdots,Y_p$, S_1, S_2 are mutually

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independent and minimal sufficient for unknown parameters of the original model, where

$$X_i \sim N(\mu_i, \alpha\sigma_1^2),\ Y_i \sim N(\mu_i, \beta\sigma_2^2) \text{ for } i=1,\cdots,p,$$
$$Y_0 \sim N(\mu_0, \gamma\sigma_2^2),$$
$$S_1/(\alpha\sigma_1^2) \sim \chi^2_m,\ S_2/(\beta\sigma_2^2) \sim \chi^2_n.$$

Here μ_0 is the unknown total mean, $\mu_1,\cdots,\mu_p$ are the unknown treatment contrasts, σ_1^2 and σ_2^2 are unknown variances such that $\sigma_1^2 < \sigma_2^2$, i.e.

$$\theta < \theta_0.$$

Also the first common mean μ_1 is, without loss of generality, considered *any treatment contrast,* and X_1 and Y_1 are regarded as the *intrablock* and the *interblock* estimators of μ_1, respectively. Then based on the statistics $(X_1,\cdots,X_p, Y_0, Y_1,\cdots,Y_p, S_1, S_2)$, we want to estimate the treatment constrast μ_1.

In the analysis of BIBD's with blocks and errors random, Yates(1940) first exhibited that there arise two independent unbiased estimators X_1 and Y_1 of treatment contrast μ_1 and derived a method for combining these two estimators so as to estimate μ_1 with greater precision than the customary intrablock estimator X_1. This analysis is called *the recovery of interblock information.* Later, other combined estimators being better than X_1 have been given by many authors[For the references, see Kubokawa(1988)]. Among these estimators, the inadmissibility of untruncated estimators has been shown by Seshadri(1966), Shah(1971), Nair(1982). More generally, Bhattacharya(1983) considered the estimators

$$\delta = X_1 + \Psi\cdot(Y_1 - X_1), \tag{1.1}$$

$$\delta_{TR} = X_1 + \min(\theta_0, \Psi)\cdot(Y_1 - X_1), \tag{1.2}$$

where

$$\Psi = \Psi(S_1, S_2, (X_i - Y_i)^2, Y_0, X_i, Y_i, i=1,\cdots,p), \tag{1.3}$$

and proved that the truncated estimator δ_{TR} dominates the untruncated one δ. All the above results were obtained for quadratic loss function. So of interest is to investigate whether the domination holds relative to general loss functions.

Lee(1987) recently demonstrated that δ is dominated by $X_1 + \min[1, \max\{0, \Psi\}]\cdot(Y_1 - X_1)$ with respect to arbitrary convex loss functions. The purpose of the paper is to prove the following theorem. Let $L(\cdot)$ be any nondecreasing function on $[0, \infty)$.

Theorem 1. Assume that $\Psi > \theta_0$ with positive probability for Ψ defined by (1.3). Then δ_{TR} in (1.2) dominates δ in (1.1) relative to any nondecreasing loss $L(|\delta-\mu_1|)$, that is,

$$E_\omega[L(|\delta_{TR}-\mu_1|)] \le E_\omega[L(|\delta-\mu_1|)] \text{ for any } \omega, \tag{1.4}$$

where $\omega=(\mu_0, \mu_1, \cdots, \mu_p, \sigma_1^2, \theta)$, unknown parameters.

The next lemmas are useful for the proof.

Lemma 1. Let $U=Y_1-X_1$ and $V=Y_1-\mu_1+a(X_1-\mu_1)$ where $a=\beta\sigma_2^2/\alpha\sigma_1^2$. Then U and V are independently distributed.

Lemma 2. Let Φ denote the distribution function of N(0, 1), and let $\bar{\Phi}=1-\Phi$. Then, for any fixed $t'>0$,

$$G(c)=\bar{\Phi}(t'-c)+\bar{\Phi}(t'+c)$$

is nondecreasing in $c>0$.

In the sequel we will use the notations in Lemma 1 and Lemma 2.

2. Proof of Theorem 1

Under the assumed model, it suffices to show that $|\delta-\mu_1|$ is stochastically larger than $|\delta_{TR}-\mu_1|$. By the conditional argument, we may regard every random variable except X_1 and Y_1 as a constant. It is also easy to see that we may assume $\mu_1=0$.

For any fixed $t>0$,

$$\begin{aligned} & P\{|\delta|>t\} \\ &=2P\{X_1+\Psi(Y_1-X_1)>t\} \\ &=2P\{(1+a)^{-1}(V-U)+\Psi U \ge t\} \\ &=2E[\bar{\Phi}\{t'+b(\theta-\Psi)U\}] \end{aligned} \tag{2.1}$$

where $t'=\dfrac{(1+a)t}{\sqrt{\alpha\sigma_1^2+\beta\sigma_2^2}\,\sqrt{a}}$ and $b=\dfrac{(1+a)}{\sqrt{\alpha\sigma_1^2+\beta\sigma_2^2}\,\sqrt{a}}$. Since U and -U are identically distributed and Ψ depends on U only through $|U|$, it follows from (2.1) that

$$\begin{aligned} & P\{|\delta|>t\} \\ &=E[\bar{\Phi}\{t'+b(\theta-\Psi)U\}+\bar{\Phi}\{t'-b(\theta-\Psi)U\}] \\ &=E[\bar{\Phi}\{t'+b(\theta-\Psi)|U|\}+\bar{\Phi}\{t'-b(\theta-\Psi)|U|\}]. \end{aligned} \tag{2.2}$$

The last identity holds because $\bar{\Phi}\{t'+b(\theta-\Psi)U\}+\bar{\Phi}\{t'-b(\theta-\Psi)U\}$ is an even function of U.

Let

$$G(b(\Psi-\theta)\,|\,U\,|\,)=\bar{\Phi}\{t'-b(\Psi-\theta)\,|\,U\,|\,\}+\bar{\Phi}\{t'+b(\Psi-\theta)\,|\,U\,|\,\}.$$

Then, from (2.2), we have

$$\begin{aligned} &P\{\,|\,\delta\,|>t\} \\ &=E[I(\Psi\leq\theta_0)\,G(b(\Psi-\theta)\,|\,U\,|\,)] \\ &+E[I(\Psi>\theta_0)\,G(b(\Psi-\theta)\,|\,U\,|\,)]. \end{aligned} \tag{2.3}$$

Since $\theta_0>\theta$, it follows from Lemma 2 that

$$\begin{aligned} &E[I(\Psi>\theta_0)\,G(b(\Psi-\theta)\,|\,U\,|\,)] \\ &\geq E[I(\Psi>\theta_0)\,G(b(\theta_0-\theta)\,|\,U\,|\,)]. \end{aligned}$$

Therefore we have, with $\Psi\wedge\theta_0=\min\{\Psi,\,\theta_0\}$,

$$\begin{aligned} &P\{\,|\,\delta\,|>t\,\} \\ &\geq E[I(\Psi\leq\theta_0)\,G(b(\Psi-\theta)\,|\,U\,|\,)] \\ &+E[I(\Psi>\theta_0)\,G(b(\theta_0-\theta)\,|\,U\,|\,)] \\ &=E[G(b(\Psi\wedge\theta_0-\theta)\,|\,U\,|\,)] \\ &=P\{\,|\,\delta_{TR}\,|>t\} \end{aligned}$$

which completes the proof of Theorem 1.

It should be remarked that results on left truncated estimator can be obtained in a similar way to Theorem 1.

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