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Search Design of Resolution III. 3 for 3^4 Factorial $^{+}$

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ABSTRACT

The basic conditions for a parallel flats fraction to be a search design of resolution III. 2 have been developed in Um(1980, 1981, 1983, 1984). A series of resolution III. 2 search design for 3^n, $n=4, 5, 6$ are presented in Um(1988). In this paper a resoultion III. 3 search design for 3^4 is presented.

1. Introduction

Suppose an experimenter knows that all three-factor and higer order interactions are zero, and that among the possible pairs of factors there are at most r pairs that interact. The designs should have a relatively small number of runs, and the procedures for estimation should be simple. One of the such designs is a resolution III design. In this design, all main effects are estimable, where two factor and higer order interactions are negligible.

Assume that at most k of the possible two factor interactions are present. The problem is to construct a design which will permit estimation of general mean and all main effects, detect the k of two-factor interactions, and estimate all degrees of freedom for these interactions. In this situation the design is said to be "Search Design of Resolution III. k". For the general basic ideas for search designs, see Srivastava(1975, 1976, 1977).

In Um(1988), it has constructed the designs which can detect at most two of two-factor interactions. In this paper a design which can detect at most three two-factor interactions will be presented.

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2. Search Design of Resolution III. 2 for 3^4

The A-matrix selected for the 3^4 factorial is

$$A=\begin{bmatrix}1 & 1 & 1 & 0\\ 1 & 2 & 0 & 1\end{bmatrix}$$

The alias sets partitioned by the A-matrix are given by

$$S_0=\{\mu\}$$
$$S_1=\{F_1,\ F_2F_3,\ F_2F_4^2,\ F_3F_4\},\ S_2=\{F_2,\ F_1F_3,\ F_1F_4,\ F_3F_4^2\}$$
$$S_3=\{F_3,\ F_1F_2,\ F_1F_4^2,\ F_2F_4\},\ S_4=\{F_4,\ F_1F_2^2,\ F_1F_3^2,\ F_2F_3^2\}$$

With the single flat $C=(c_1,\ c_2)$ the defining vectors of ACPM where columns are associated with effects in the same ordering as listed in the alias sets are given by

$$C_1^*=(0\quad 2c_1\quad 2c_2\quad c_1+c_2),\qquad C_2^*=(0\quad 2c_1\quad c_2\quad c_1+2c_2),$$
$$C_3^*=(0\quad 2c_1\quad 2(c_1+c_2)\quad 2c_1+c_2),\ C_4^*=(0\quad 2c_2\quad 2(c_1+c_2)\quad c_1+2c_2).$$

There are only eight equivalence classes of C-matrix with two rows and three columns. The number of those classes are enumerated in Um(1981, 1984). In order to find a search design it is enough to consider only one element in each equivalence class.

For $C=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 2\end{bmatrix}$, the following ACPM are produced.

$$P_1=\begin{bmatrix}e & e & e & e\\ e & e & (021) & (012)\\ e & (021) & (012) & e\end{bmatrix},\quad P_2=\begin{bmatrix}e & e & e & e\\ e & e & (012) & (021)\\ e & (021) & (021) & (021)\end{bmatrix},$$

$$P_3=\begin{bmatrix}e & e & e & e\\ e & e & (012) & (012)\\ e & (021) & e & (012)\end{bmatrix},\quad P_4=\begin{bmatrix}e & e & e & e\\ e & (021) & (021) & (021)\\ e & (012) & e & (021)\end{bmatrix},$$

For the (0, 1) detection matrix, see Um(1988).

The treatment combination for the above C can be obtained in flats of size nine by solution to

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The treatment combinations are displayed below:

Flat 1	Flat 2	Flat 3
0 0 0 0	0 0 0 1	0 0 1 2
0 1 2 1	0 1 2 2	0 1 0 0
0 2 1 2	0 2 1 0	0 2 2 1
1 0 2 2	1 0 2 0	1 0 0 1
1 1 1 0	1 1 1 1	1 1 2 2
1 2 0 1	1 2 0 2	1 2 1 0
2 0 1 1	2 0 1 2	2 0 2 0
2 1 0 2	2 1 0 0	2 1 1 1
2 2 2 0	2 2 2 1	2 2 0 2

3. Search Design of Resolution III. 3 for 3^4

In section 2, the resolution III. 2 design was constructed for the 3^4 factorial. The design is almost resolution III. 3. By this we mean that several combinations of three pairs of interactions can be detected and estimated. However, there are some combinations for which additional runs would be required to complete the estimation.

Since there are four main effects, there are six two-factor interactions. Therefore, $\binom{6}{0}+\binom{6}{1}+\binom{6}{2}+\binom{6}{3}=42$ detection vectors can be constructed. Table 1 shows the (0, 1) detection matrix for the 3^4 factorial where $k=3$. The first row denotes the difference of the ith row and i' row, and the second row represents the subscripts of ACPM. The column 4-11 denotes the subscripts of two-factor interactions.

There are three cases which have the identical (0, 1) detection vector:

(1) $(F_1F_2,\ F_1F_4)$, $(F_1F_2,\ F_1F_3,\ F_1F_4)$

(2) $(F_1F_2,\ F_2F_4)$, $(F_1F_2,\ F_2F_3,\ F_2F_4)$

(3) $(F_1F_4,\ F_2F_4)$, $(F_1F_4,\ F_2F_4,\ F_3F_4)$

Case 1. From ACPM the following submatrices which are full rank are obtained:

$$\begin{array}{c} P_1: F_1 \\ \begin{bmatrix} e \\ e \\ e \end{bmatrix} \end{array}, \quad \begin{array}{ccc} P_2: F_2 & F_1F_3 & F_1F_4 \end{array} \begin{bmatrix} e & e & e \\ e & e & (012) \\ e & (021) & (021) \end{bmatrix}, \quad \begin{array}{ccc} P_3: F_3 & F_1F_2 & F_1F_4 \end{array} \begin{bmatrix} e & e & e \\ e & e & (021) \\ e & (021) & e \end{bmatrix},$$

$$\begin{array}{ccc} P_4: F_4 & F_1F_2 & F_1F_3 \end{array} \begin{bmatrix} e & e & e \\ e & (021) & (021) \\ e & (012) & e \end{bmatrix}$$

Therefore, all effects of the two-factor interactions F_1F_2, F_1F_3, and F_1F_4 are estimable.

Case 2.

$$\begin{array}{ccc} P_1: F_1 & F_2F_3 & F_2F_4 \end{array} \begin{bmatrix} e & e & e \\ e & (021) & (012) \\ e & (012) & e \end{bmatrix}, \quad \begin{array}{c} P_2: F_2 \\ \begin{bmatrix} e \\ e \\ e \end{bmatrix} \end{array}, \quad \begin{array}{ccc} P_3: F_3 & F_1F_2 & F_2F_4 \end{array} \begin{bmatrix} e & e & e \\ e & e & (012) \\ e & (021) & (021) \end{bmatrix}$$

$$\begin{array}{ccc} P_4: F_4 & F_1F_2 & F_2F_3 \end{array} \begin{bmatrix} e & e & e \\ e & (021) & (021) \\ e & (021) & (021) \end{bmatrix}$$

All effects of the two-factor interactions F_1F_2, F_2F_3, and F_2F_4 are estimable.

Table 1. The (0, 1) Detection Matrix for the 3^4 Factorial with K=3

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

				1-2				1-3				2-3			
				1	2	3	4	1	2	3	4	1	2	3	4
1		MAIN		0	0	0	0	0	0	0	0	0	0	0	0
2			12	0	0	0	1	0	0	1	1	0	0	1	1
3			13	0	0	0	1	0	1	0	0	0	1	0	1
4			14	0	1	1	0	0	1	0	0	0	1	1	0
5			23	0	0	0	1	1	0	0	1	1	0	0	0
6			24	1	0	1	0	1	0	1	0	1	0	0	0
7			34	1	1	0	0	0	1	0	0	1	0	0	0
8		12	13	0	0	0	1	0	1	1	1	0	1	1	1
9		12	14	0	1	1	1	0	1	1	1	0	1	1	1
10		12	23	0	0	0	1	1	0	1	1	1	0	1	1
11		12	24	1	0	0	1	1	0	1	1	1	0	1	1
12		12	34	1	1	0	1	0	1	1	1	1	0	1	1
13		13	14	0	1	1	1	0	1	0	0	0	1	1	1
14		13	23	0	0	0	1	1	1	0	1	1	1	0	1
15		13	24	1	0	1	1	1	1	1	0	1	1	0	1
16		13	34	1	1	0	1	0	1	0	0	1	1	0	1
17		14	23	0	1	1	1	1	1	0	1	1	1	1	0
18		14	24	1	1	1	0	1	1	1	0	1	1	1	0
19		14	34	1	1	1	0	0	1	0	0	1	1	1	0
20		23	24	1	0	1	1	1	0	1	1	1	0	0	0
21		23	34	1	1	0	1	1	1	0	1	1	0	0	0
22		24	34	1	1	1	0	1	1	1	0	1	0	0	0
23	12	13	14	0	1	1	1	0	1	1	1	0	1	1	1
24	12	13	23	0	0	0	1	1	1	1	1	1	1	1	1
25	12	13	24	1	0	1	1	1	1	1	1	1	1	1	1
26	12	13	34	1	1	1	1	1	1	1	1	1	1	1	1
27	12	14	23	0	1	1	1	1	1	1	1	1	1	1	1
28	12	14	24	1	1	1	1	1	1	1	1	1	1	1	1
29	12	14	34	1	1	1	1	0	1	1	1	1	1	1	1
30	12	23	24	1	0	1	1	1	0	1	1	1	0	1	1
31	12	23	34	1	1	0	1	1	1	1	1	1	0	1	1
32	12	24	34	1	1	1	1	1	1	1	1	1	0	1	1
33	13	14	23	0	1	1	1	1	1	0	1	1	1	1	1
34	13	14	24	1	1	1	1	1	1	1	0	1	1	1	1
35	13	14	34	1	1	1	1	0	1	0	0	1	1	1	1
36	13	23	24	1	0	1	1	1	1	1	1	1	1	0	1
37	13	23	34	1	1	0	1	1	1	0	1	1	1	0	1
38	13	24	34	1	1	1	1	1	1	1	0	1	1	0	1
39	14	23	24	1	1	1	1	1	1	1	1	1	1	1	0
40	14	23	34	1	1	1	1	1	1	0	1	1	1	1	0
41	14	23	34	1	1	1	0	1	1	1	0	1	1	1	0
42	23	24	34	1	1	1	1	1	1	1	1	1	0	0	0

COMPARISON OF THE ROWS OF (0, 1) DETECTION MATRIX

1	ROW NUMBER	9	AND ROW NUMBER	23	ARE EQUAL
2	ROW NUMBER	11	AND ROW NUMBER	30	ARE EQUAL
3	ROW NUMBER	18	AND ROW NUMBER	41	ARE EQUAL

Case 3.

$$P_1: \begin{matrix} F_1 & F_2F_4 & F_3F_4 \end{matrix} \quad P_2: \begin{matrix} F_2 & F_1F_4 & F_3F_4 \end{matrix} \quad P_3: \begin{matrix} F_3 & F_1F_4 & F_2F_4 \end{matrix}$$

$$\begin{bmatrix} e & e & e \\ e & (021) & (012) \\ e & (012) & e \end{bmatrix}, \begin{bmatrix} e & e & e \\ e & (012) & (021) \\ e & (021) & (021) \end{bmatrix}, \begin{bmatrix} e & e & e \\ e & (021) & (012) \\ e & e & (012) \end{bmatrix}$$

$$P_4: F_4$$

$$\begin{bmatrix} e \\ e \\ e \end{bmatrix}$$

All effects of the two-factor interactions F_1F_4, F_2F_4, and F_3F_4 are estimable.

The resolution III. 2 design for the 3^4 factorial is also a resolution III. 3 design except for the four cases which are shown in Table 2. The table also contains the alias set for which the corresponding ACPM is not full rank.

In each of the four cases listed in Table 2 one of the ACPM is less than full rank and all others are full rank. It is a simple procedure to adjoin three additional observations to the design such that the ACPM in question is brought to full rank. Choose one of the other alias sets which is of full rank and alias those effects with μ in a flat of size 3. In general, the new observations are of the form

$$\begin{bmatrix} 0\ 0 \cdots 0\ 1\ 0 \cdots 0 \\ A \end{bmatrix} \underline{t} = \begin{bmatrix} 0 \\ \underline{c} \end{bmatrix}$$

Table 2. Combinations Which are not Full Rank for the 3^4 Factorial

	Combinations	Alias Set
(1)	$(F_2F_3,\ F_2F_4,\ F_3F_4)$	S_1
(2)	$(F_1F_3,\ F_1F_4,\ F_3F_4)$	S_2
(3)	$(F_1F_2,\ F_1F_4,\ F_2F_4)$	S_3
(4)	$(F_1F_2,\ F_1F_4,\ F_2F_3)$	S_4

The components of $\underline{c}$ can be chosen so the ACPM which is less than full rank is brought to full rank. The same defining vector for the ACPM as for the flats of size nine can be used. This will be illustrated for the four cases in Table 2.

Case 1. Defining vector for P: (0 $2c_1$ $2c_2$ c_1+c_2).

New observations:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \underline{t} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

t_1	t_2	t_3	t_4
0	0	2	0
1	0	1	2
2	0	0	1

ACPM: F_1 F_2F_3 $F_2F_4^2$ F_3F_4

$$P_1 = \begin{bmatrix} e & e & e & e \\ e & e & (021) & (012) \\ e & (021) & (021) & E \\ e & e & (012) & (021 \end{bmatrix}$$

Case 2. Defining vector: (0 $2c_1$ c_2 c_1+c_2)

New observations:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \underline{t} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

t_1	t_2	t_3	t_4
0	0	2	0
1	0	1	2
2	0	0	1

ACPM: F_1 F_1F_3 F_1F_4 $F_3F_4^2$

$$P_2 = \begin{bmatrix} e & e & e & e \\ e & e & (012) & (012) \\ e & (021) & (021) & (021) \\ e & e & (012) & e \end{bmatrix}$$

Case 3. Defining vector: (0 $2c_1$ $2(c_2+c_2)$ $2c_1+2c_2$)

New observations:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \qquad \underline{t} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \qquad \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \end{array}$$

ACPM: F_3 F_1F_2 $F_1F_4^4$ F_2F_4

$$P_3 = \begin{bmatrix} e & e & e & e \\ e & e & (021) & (012) \\ e & (021) & e & (012) \\ e & (012) & (021) & e \end{bmatrix}$$

Case 4. Defining vector: $(0 \quad 2c_2 \quad 2(c_1+c_2) \quad 2c_1+2c_2)$

New observations:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \qquad \underline{t} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 2 & 2 & 0 & 0 \end{array}$$

ACPM: F_4 $F_1F_2^2$ $F_1F_3^2$ $F_2F_3^2$

$$P_4 = \begin{bmatrix} e & e & e & e \\ e & (021) & (021) & (021) \\ e & (012) & e & (021) \\ e & e & (021) & (012) \end{bmatrix}$$

In this manner, we can construct a design which can detect at most three two-factor interactions for n=5, 6. For the 3^5 factorial, the resolution III. 3 design produces 55 identical rows of the detection matrix when all combinations of three pairs of interactions are included. Most of these are resolvable. For the 3^6 factorial, the resolution III. 3 design produces 154 identical rows of the detection matrix when all combinations of three pairs of interactions are included. All of these are also resolvable.

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