An Expert Finite Element Discretization for Time-Dependent Structural Problems

시간 종속 구조응력해석을 위한 전문가 유한요소 모델링

朱 寬 正*
Joo, Kuan Jung

요 약

본 논문에서는 시간 종속 하중을 받는 대형 전산구조응력 해석 문제를 위한 유한요소 모델링 기법이 소개된다. 유한요소모델링의 분할기준은 문제에 대한 해석 결과에 대한 오차선정에 근거를 둔다. 이 오차산정은 해석결과치에 의한 잔유 에너지의 크기를 유한요소별로 산정한다. 이의 시간존장 구조물에 대한 응용은, 구조연속체의 Ritz 고유진동 모드를 계산하고 이들 진동 모드 중에서 저주파에 상응하는 진동모드에 대해 잔유 에너지의 크기가 구조체 전체영역에서 평형을 유지하도록 유한요소 모델링을 수행한다. 마지막으로, 여기서 제안된 알고리즘이 몇 예제들을 통해서 검증된다.

SUMMARY

A finite element technique for the time dependent large structural problems is presented. It is based on the error estimation for the bases of solution spaces. An *a-posteriori* energy norm of residual error serves as the the error indicator. Mode shapes which are calculated by scaling the Ritz vectors are applied to discretize the continuous spatial domain. Finally, the performance of the proposed methods is demonstrated by solving simple examples.

1. INTRODUCTION

Due to the large size and geometrical complexity of engineering problems, it is generally not easy to solve them with minimum level of computational effort. It is true that the discretization of the finite element model for a particular problem is largely dependent upon experience and intuition. Recently, finite element discretization techniques based on reliable error analysis and mesh refinement feedback procedures have emerged as an efficient computational methods[e. g. 1].

For time dependent problems, the discretization for the finite element analysis can be carried out

* 정회원, 三星電機(株) 綜合研究所 CAD/CAM 室, 공학박사

이 논문에 대한 토론은 1990년 12월 31일까지 본 학회에 보내주시면 1991년 6월호에 그 결과를 게재하겠습니다.

in space with/without considering time variables. Since the topology of finite elements is function of position and time, the mesh configuration at each time step can be obtained by minimizing a specified error indicator. The first approach for the spatial discretization is the moving discretiz ation technique which optimizes positions of the nodal points of discretized mesh without increasing the number of equations. In the second approach the previously refined nodal points are not reloc-ated and the number of equations is increased/ decreased at each time step. In the above appro-aches, however, there are several drawbacks. An error analysis is required at each time step. Esp-ecially, the first approach has penalties in the respects of element distortion and element cont-inuity ; and the second approach requires a special algorithm to handle data structure. The third approach was developed to avoid those weakness ; the discretization is completed before time inte-grati on. The mesh is required sufficiently enough to capture the frequencies which are high comp-aring with the frequencies of external loading.

To control the finite element discretization, a proper error estimation is necessary. In general, sources of error can be categorized into three parts : mathematical modeling error, discretization error and solution error. The mathematical modeling error results from the medelings of materials, boundary conditions and loading conditions. The discretization error can be contributed to the spatial and the time discretizations. The temporal portion is associated with the numerical errors. The spatial error contribution results from the element size and the element type, and thus, it can be estim-ated based on sources of error such as defects in the local governing equations, stationary conditions for the functionals of field variables, etc. The solution error consists of the round-off error and the cancellation error during algorithmic solution process.

In this paper, the spatial discretization is perf-ormed in the context of adaptive mesh refinement. In the next section, mathematical basis of the adaptive mesh refinement is described. Section 3 presents development of an *a−posteriori* error indicator in energy norm. Residuals inside and on the bounbary of each element, resulted from the finite element spatial discretization, are considered. The derivation of the energy norm of residual error is presented. In Section 4, numerical examples are presented to illustrate the performance of the proposed methods. As the basis of solution spac-e,the 'dervied'Rit vectors[2, 3] are used. Concluding remarks are addressed in Section 5.

2. BASIS OF ADAPTIVE FINITE ELEMENT

Consider C° −finite elements and let its sub-space be $V_h(\Omega)\subset C^\circ(\underline{\Omega})$. Here $\underline{\Omega} = \Omega \cup \Gamma$ where Ω denotes a continuous interior domain and Γ the boundary. The finite element subspace $V_h(\Omega)$ with maximum size h can be constructed on $\overset{N}{\underset{e=1}{\cup}} \underline{\Omega}$ where Ω_e is a convex open domain. Then it has a following structure in the context of hierarchical type of adaptive element approximation

$$Vh_1{}^1(\Omega)\subset\cdots \subset Vh_{i+1}{}^{P_i}(\Omega)\subset\cdots Vh_{i+1}{}^{p_i+1}(\Omega) \subset\cdots$$

where $Vh_iP_i(\Omega)$ is a subspace based on the P_i degree of interpolation basis function constructed hierarchically and the maximum size h_i of Ω_e which is discretized from Ω is hierarchical fashion. For example, $V_{h1}{}^1(\Omega)$ is based on a linear basis function $N_i{}^{(1)}(\Omega)$ (ξ, η, ζ) in the original mesh and $Vh_2^2(\Omega)$ is based on a hierarchic quadratic basis function $N_i{}^{(2)}(\xi, \eta, \zeta)$ in the first level of hierarc-hical h−refinement.

A weak from of the undamped dynamics pro-blems can be expressed as in ref.[4] :

$$(\rho\ddot{u}, \boldsymbol{\eta})_\Omega + a(\mathbf{u}, \boldsymbol{\eta})_\Omega = (\mathbf{p}, \boldsymbol{\eta})_\Omega + (\mathbf{t}, \boldsymbol{\eta})\Gamma_t \qquad (2.\ 1)$$

for $arb.\ \boldsymbol{\eta} \in W^{m,p}(\Omega)$

where $W^{m,p}$ is the Sobblev space [e. g. 5] defined by

$$W^{m,p}(\Omega) = \{\boldsymbol{\eta} \mid \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \in L^P(\Omega)$$

$$\text{for } 0 \leqq \| \alpha_1 + \cdots \alpha_n \| \leqq m\} \qquad (2.\ 2)$$

in which $L^P(\Omega)$ is the linear space of equivalence class of measurable functions $\boldsymbol{\eta}$ such that

$$\int_\Omega | \boldsymbol{\eta}(x) |^P d\mathbf{x} < \infty \qquad (2.\ 3)$$

Since $W^{m,p}(\Omega)$ is an infinite dimensional space it is not possible to compute the error exactly. Thus, the error is computed when the approximation space is enlarged from $Vh_i^{P_i}$ to $Vh_{i+1}^{P_{i+1}}$.

Considering two levels of the refinement process, the weak forms of each case can be written as

$$(\rho\ddot{\mathbf{u}}_{(1)}, \boldsymbol{\eta}_{(1)})\Omega + a(\mathbf{u}_{(1)}, \boldsymbol{\eta}_{(1)})_\Omega$$

$$= (\mathbf{p}, \boldsymbol{\eta}_{(1)})_\Omega + (\mathbf{t}, \boldsymbol{\eta}_{(1)})\Gamma_t \qquad (2.\ 4)$$

for $arb.\ \boldsymbol{\eta}_{(1)} \in Sh_i^{P_i}(\Omega)$

and

$$(\rho\ddot{\mathbf{u}}_{(2)}, \boldsymbol{\eta}_{(2)})\Omega + a(\mathbf{u}_{(2)}, \boldsymbol{\eta}_{(2)})_\Omega$$

$$= (\mathbf{p}, \boldsymbol{\eta}_{(2)})_\Omega + (\mathbf{t}, \boldsymbol{\eta}_{(2)})\Gamma_t \qquad (2.\ 5)$$

for $arb.\ \boldsymbol{\eta}_{(2)} \in S_{h_{i+1}}^{p_{i+1}}(\Omega)$

where $\mathbf{u}_{(2)}$ is closer to the exact value $\mathbf{u}$ than $\mathbf{u}$ than $\mathbf{u}_{(1)}$. Hence the error pertaining to the equation(2.4) is less than that of equation(2. 5). The error in the approximation space, $(Vh_{i+1}^{P_{i+1}} - Vh_i^{P_i})$, is equal to the exact error, $\mathbf{e}=\mathbf{u}-\mathbf{u}_{(1)}$, projected onto the space $(Vh_{i+1}^{P_{i+1}} - Vh_i^{P_i})$. Our goal is to calculate the error in the space of $(Vh_{i+1}^{P_{i+1}} - Vh_i^{P_i})$ and to minimize it with less degrees of freedom.

The error estimates can be expressed, assuming the regularity of the exact solution is sufficiently large, in the from of

$$\| \mathbf{e}_{m,\Omega} \| = ch2^{-(K+2-m)} \| \mathbf{e} \|_{r,\Omega} \qquad (2.\ 6)$$

where k is the order of the elements, m is the derivatives in the weak form, and c is a constant which is dependent upon P and Ω.

3. ENERGY NORM OF RESIDUAL ERROR

In the continuum body, the momentum balance is given by

$$\text{div}\ \boldsymbol{\sigma} + \mathbf{b} = \rho\ddot{\mathbf{u}} \quad \text{in } \Omega \qquad (3.\ 1)$$

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} = \underline{\mathbf{t}} \quad \text{on } \Gamma_t \qquad (3.\ 2)$$

where

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \text{ and } \Gamma = \Gamma_t \cup \Gamma_u \qquad (3.\ 3)$$

with the kinematics given by

$$\boldsymbol{\varepsilon} = \nabla^S \mathbf{u} \text{ in } \Omega \qquad (3.\ 4)$$

$$\mathbf{u} = \underline{\mathbf{u}} \quad \text{on } \Gamma_u \qquad (3.\ 5)$$

in which

$$\nabla^S \mathbf{u} = 1/2[\nabla \mathbf{u}(\nabla \mathbf{u})^T \qquad (3.\ 6)$$

and the constitutive equation given by

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} \text{ in } \Omega \qquad (3.\ 7a)$$

in which

$$\psi = \psi(\nabla^S \mathbf{u}) \qquad (3.\ 7b)$$

A weak from of the above boundary value equations can be constructed as a following equation

$$G(\mathbf{u}, \boldsymbol{\eta}) = \int_\Omega \frac{\partial\psi}{\psi\boldsymbol{\epsilon}} : \nabla^S \boldsymbol{\eta} d\Omega$$

$$+ \int_\Omega (\rho\ddot{\mathbf{u}} - \mathbf{b}) \cdot \boldsymbol{\eta}\, d\Omega - \int_{\Gamma_t} \underline{\mathbf{t}} \cdot \boldsymbol{\eta}\, d\Gamma = 0 \qquad (3.\ 8)$$

where η are arbitrary functions satisfying appropriate boundary such that

$$\eta \in V_0 \equiv \{\eta : \Omega \rightarrow \mathbf{R}^3 \ \|\boldsymbol{\eta}\|_{\Gamma_u} = \mathbf{0}\} \qquad (3.\ 9)$$

Integrating equation (3. 8) by parts with the introduction of discretization and assuming C° – element, equation (3. 8) leads to

$$\sum^e \int_{\Omega^e} (\text{div}\,\boldsymbol{\sigma} + \mathbf{b} - \rho\ddot{\mathbf{u}}) \cdot \boldsymbol{\eta}\, d\Omega$$

$$-2\sum^e \int_{S_e} (\mathbf{t} - \mathbf{t}_{avg}) \cdot \boldsymbol{\eta}\, d\Gamma \qquad (3.\ 10)$$

$$+ \sum^e \int_{\Gamma_t^e} (\underline{t} - \boldsymbol{\sigma}\mathbf{n}) \cdot \boldsymbol{\eta}\, d\Gamma = 0 \qquad (3.\ 10)$$

where $S^e \bigcup_e \Gamma^e - \Gamma$ and $\mathbf{t}_{avg}$ is the half magnitude of the stresses discontinuity on the element boundary, S^e.

Since the exat solution is generally not known, the following expression, can be taken as the element error indicator :

$$\|\mathbf{e}\|^2_{E,\ \underline{\Omega}^e} = \|\mathbf{e}\|^2_{E,\ \Omega^e} + \|\mathbf{e}_\Gamma\|^2_{E,\ \Gamma^e} \qquad (3.11)$$

where

$$\|\mathbf{e}_\Omega\|^2_{E,\ \Omega^e} = \frac{1}{n_2 K} \Omega^e \int_{\Omega^e} (\mathbf{r}^2_{\Omega x} + \mathbf{r}^2_{\Omega y}) d\Omega \qquad (3.\ 12)$$

$$\|\mathbf{e}_\Gamma\|^2_{E,\ \Omega^e} = \frac{1}{n_2 K} \Gamma^e \int_{\Gamma^e} (\triangle t_x^2 + \triangle t_y^2)\, d\Gamma \qquad (3.\ 13)$$

In the above equation, n denotes the order of shape function ; k is the material constant ; $\mathbf{r}_{\Omega x}$ and $\mathbf{r}_{\Omega y}$ denote the residuals in x- and y-direction, respectively ; $\triangle \boldsymbol{t}_x$ and $\triangle \boldsymbol{t}_y$ denote the half of the jump of truncation force on element boundary S^e in x- and y- direction, respectively.

4. NUMERICAL EXAMPLES

In this section, two numerical examples are presented to demonstate the performance of the proposed error measure. The first example is considered to illustrate the efficiency of the error measure and the second is to show its effectiveness with the use of 'derved' Ritz vectors.

The spatial discretization is based on a simple criterion, which reflects the uniformity of residual error over all elements. Since the residual forces of all elements are self equilibrated in the structural system, normalization with respect to the volume(or element size) is not considered. Instead, the energy norm of each element is normalized by the maximum of all elements. The refinement process continues until errors of all elements are lowered to a given equilibration level. For the dynamics problems, the discretization procedure is applied to each mode shape which is calculated by scaling the Ritz vectors with weighting factors. The selecting and scaling procedure of Ritz vectors is summarized in Ref. [2]. At each stage of refinement the discretization procedure is processed and the final set of Ritz vectors are used to complete the dynamic analysis.

(a) *Example* 1 : The proposed error estimator is investigated from the point of assessing accuracy of the solution. For this purpose, only the static mode shape is considered. As illustrated in Figure 1, a dovetail joint, whose material is composite, is subjected to an external moment, M_z The example was modeled with four-node, plane strain, quadrilateral elements and the in-plane force as shown. The refined mesh at each level is shown in Figure 2. It is apparent from the error norms tabulated in Table 1 that the traditional error estimation which considers only the interelement traction jump is not an appropriate representation of the total residual error. As indicated in Figures 3 and 4, consideration of material factor (k) can be effectively used for the problems with composite materials.

(b) *Example* 2 : A plate with a hole, subjected to triangular pulse loading and modeled with two nine-node elements as shown in Figure 5 is considered. The effect of the static deflection is included when Ritz vectors are calculated. All levels of mesh refinements are shown in Figures 6(a), (b), (c), (d) and (e). Table 2 indicates how each Ritz vector contributes to the mesh refinement. Table 3 gives a comparison of response resulting from the adaptive refinement and the uniform refinement.

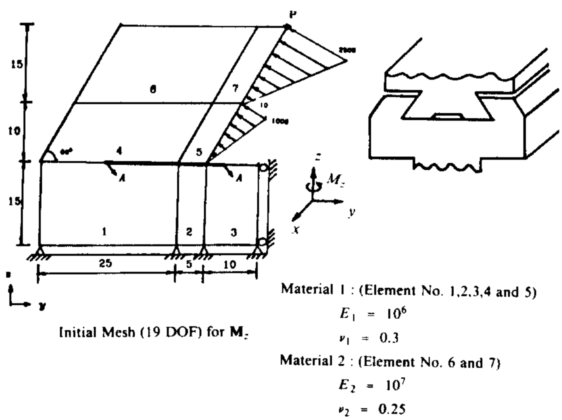


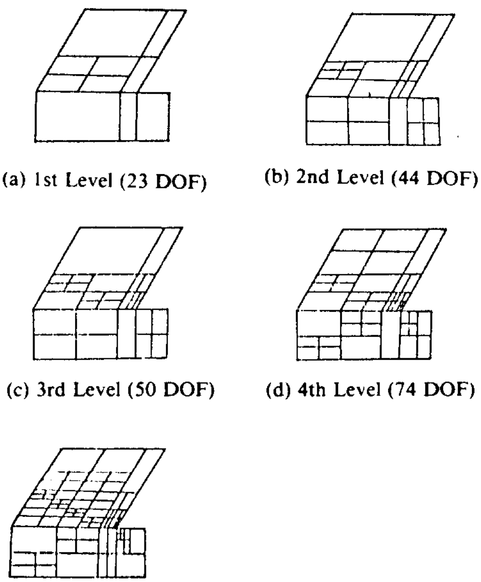
Figure 1.

Table 1. Error for Initial Mesh

m	$\frac{\|e\|_{E,m}}{\max_m \|e\|_{E,m}}$	$\|e\|_{E,m}$	$\|e_\Omega\|_{E,m}$	$\|e_r\|_{E,m}$
1	0.66	46.55	39.13	25.22
2	0.23	16.27	6.29	15.00
3	0.47	32.57	21.97	24.04
4	1.00	69.67	59.16	36.78
5	0.50	34.57	15.04	31.14
6	0.39	27.50	21.88	16.66
7	0.15	10.56	0.77	10.53

Table 2. Contribution of Ritz vectors to mesh refinement

Third Level	
No. of Ritz Vectors	Element Number at Second Level
1	3 10
2	3 10
3	3 10
4	3 10
6	3 10
8	3 10
Fourth Level	
No. of Ritz Vectors	Element Number at Second Level
1	1 6
2	5 6 8
3	1 6 6
4	2 6 8
6	6
Fifth Level	
No. of Ritz Vectors	Element Number at Second Level
1	1 3 10 12 13 18 20 22 27 29 31
2	9 10 12 13 19 21 22 22 27 29 31
3	2 4 10 12 13 19 22 29 30 31
4	10 12 13 15 21 27 29 30 31
6	9 10 11 12 26 27 29 30 31
8	9 10 11 13 27 29 30 31



(a) 1st Level (23 DOF) (b) 2nd Level (44 DOF)

(c) 3rd Level (50 DOF) (d) 4th Level (74 DOF)

(e) 5th Level (132 DOF)

Figure 2.

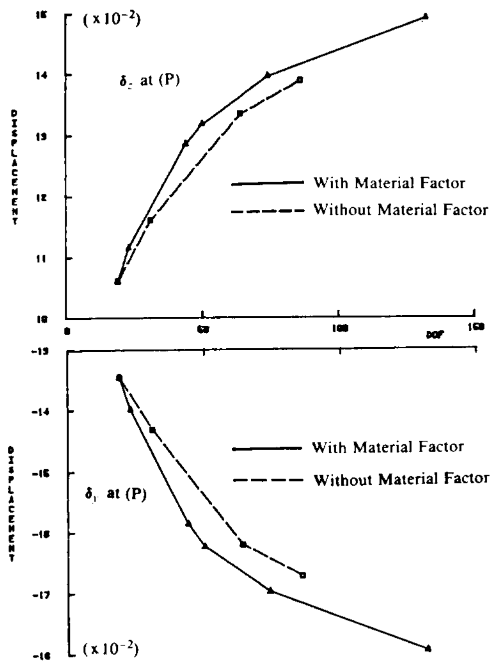


Figure 3. Convergence of Displacement vs. D.O.F.

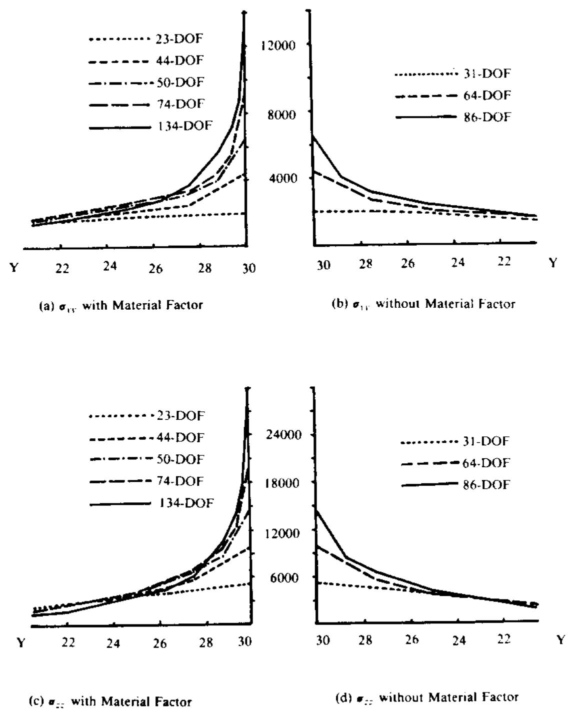


Figure 4. convergence of Stress vs. D.O.F.(Section A-A)

5. CONCLUDING REMARKS

A practical and expert discretization for the finite element analysis of structural problems is presented. The same discretization principle as presented in this paper can be applied to field problems in encountered in applied engineering science. However, since a smallness of error under one error measure does not necessarily imply that the error estimate under all of the other measures will also be small, the determination of a reliable error measure depends on the problem to be solved and on the objectives to be estimated.

For the fluid-structure interaction problems[6], it is common to have a problem which is large in the fluid region comparing with the structure region and long in the range of time with respect to time step size. In those problems, the proposed error estimator can effectively be utilized.

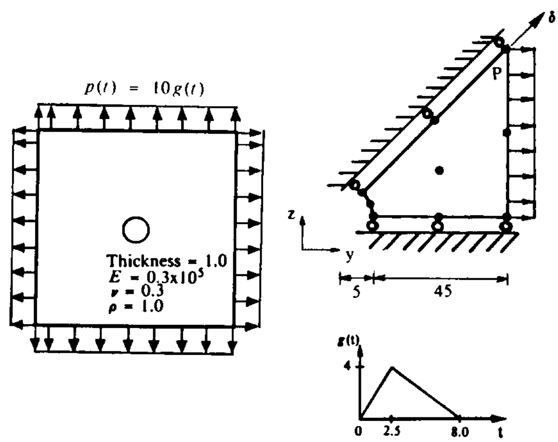


Figure 5.

6. REFERENCES

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Table 3. Response of uniform and adaptive refinement at first peak (at $t=3.0$)

Adaptive Refinement		Uniform Refinement	
degree of freedom	δ_p	degree of freedom	δ_p
88	0.0969	40	0.0924
132	0.0984	144	0.0971
262	0.0998	–	–

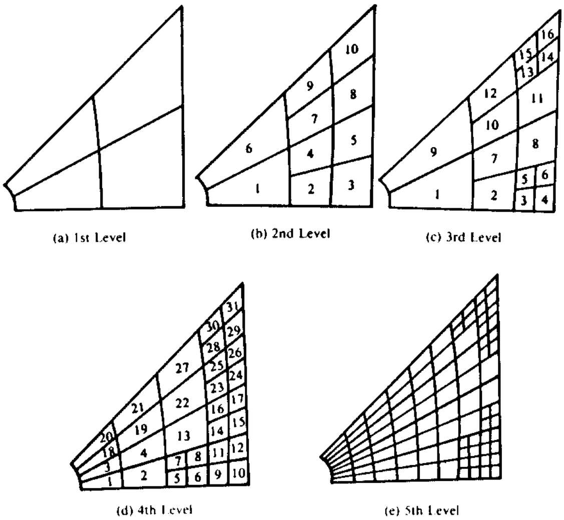

Figure 6. Mesh Refinement

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〈접수일자 : 1990. 7. 20〉