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Tighting Throughput Bounds for Finite Tandem Queues via Duality

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Abstract

This note shows that the throughput bounds for two-stage tandem queueing systems with finite buffers, obtained by the existing methods, can significantly be tightened via duality.

Keywords : Tandem queues, finite buffers, throughput bounds, duality

1. Introduction

The problem of obtaining throughput bounds for tandem queues with finite buffers has recently received a good deal of research attention. However, only a few significant studies have been reported to our knowledge, which are all confined to two-stage cases. Shanthikumar and Jafari[7] employed a method of decomposition, and Van Dijk and Lamond[8] used an interesting product form modification approach.

There is other (important) line of studies which directly addresses the more general open queueing networks containing our tandem queue as its special cases. Onvural and Perros[6] used some equivalencies between open and closed queueing networks with finite buffers.

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Bell[1] pointed out that the throughputs approximated by several decomposition approaches are even higher than the upper bound he theoretically derived.

This note is to show that the application of the concept of duality to the existing throughput bounding methods can significantly tighten the bounds for two-stage tandem queues.

The underlying idea of duality is not new. It was Gordon and Newell[2] who introduced the duality of cyclic queueing system with restricted queue length. Yamazaki et al.[9] proved that a two-stage blocking queueing system and its reversed counterpart, which is indeed the dual of the original system have the same throughput. Melamed[4] dealt with the same system as Yamazaki et al., specifically mentioning the duality concept. This concept was also used by Lamond[3] as a symmetry property for reducing the amount of computation of the blocking probabilities for two queues in tandem.

2. Duality

Consider a two-stage queueing system with a single server and a finite buffer at each stage as shown in Fig. 1, which is the same as that considered in [8]. The external arrivals at the first stage is a Poisson process with mean rate λ. The service times are independent of each other and those at the i th stage ($i=1, 2$) are exponentially distributed with mean rate μ_i. The buffer capacity of the i th stage including the service space is equal to N_i. An arriving customer who finds the first stage having N_1 customers is turned away and has no influence on the system.

The first server is not allowed to start service (or blocked) until a space is available at the second buffer. But the position in front of the server is occupied when the server is blocked. This blocking mechanism is referred to as type 2. 2 by Onvural and Perros[5]. Note that in any exponential queueing networks this blocking mechanism is equivalent to type 3. 1 because of the memoryless property of exponential services (see [5]).

The system will be referred to as the primal system with configuration of $(\lambda, N_1, \mu_1, N_2, \mu_2)$. The exact throughput is defined by λ^*.

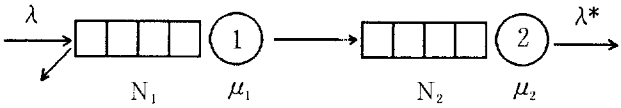

Fig. 1. A Two-Stage Tandem Queueing System with Finite Buffers

The dual system is obtained from the primal one by reversing the flow of customers and interchanging the behaviors of the stages[4]. That is, for the primal configuration $(\lambda, N_1, \mu_1, N_2, \mu_2)$, let us define the dual configuration $(\mu_2, N_2, \mu_1, N_1, \lambda)$. Fig. 2 shows the dual system with parameters corresponding to the primal one.

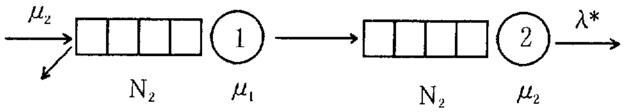

Fig. 2. The Dual Tandem Queueing System with Finite Buffers

The following theorem of duality will be used for tightening the throughput bounds.

Theorem 1. (given as Lemma 6. 5. 1 in [3]) *The primal and dual systems have the same throughput.*

3. Tightening Throughput Bounds via Duality

For exposition brevity, Onvural and Perros[6], Shanthikumar and Jafari[7], and Van Dijk and Lamond[8] will be simply denoted by O-P, S-J, and VD-L respectively. Also the throughput bounds obtained from the primal and dual systems will be referred to as the primal and dual bounds respectively.

In general, the primal and dual bounds obtained by any bounding method are different from each other, except for the case of $N_1 = N_2$ and $\lambda = \mu_2$, yielding the identical primal and dual systems. It is then expected that a bound (upper or lower) obtained by any bounding method can further be tightened by selecting the stronger one between the primal and dual bounds.

It is easy to observe this tightening effect on the following existing bounds : Bell's upper bound[1], S-J's upper and lower bounds, and VD-L's lower bound. However in the following, we show that this method doesn't generate any tightening effect on O-P's upper and lower bounds and VD-L's upper bounds.

Proposition 1. *The O-P's primal and dual bounds (upper and lower) are equal.*

Proof. The tailoring of O-P's method to our tandem queue results in a cyclic queueing system where any ordering of the given set of nodes yields the same throughput.

Proposition 2. *The O-P's and VD-L's throughput upper bounding methods for two-stage tandem queues are equivalent.*

Proof. It can be easily shown that both throughput upper bound models have the same rate matrix.

From Propositions 1 and 2, we have the following :

Proposition 3. *The VD-L's primal and dual upper bounds are equal.*

4. Numerical Examples

Computational experiment is conducted with the examples used in [6, 7, 8] to demonstrate the tightening effect of our dual bounds.

Table I lists six throughput upper bounds : four by the existing methods[1, 6, 7, 8], and two dual bounds obtained from Bell's and S-J's methods. Bell's dual bound is strictly better than its primal (original) counterpart for 8 out of 16 cases, while S-J's dual for 9 out of 16 cases. All these cases seem to share the common phenomenon that the bottleneck of the throughput is formed on the second phase (i. e., $N_1 > N_2$ or $\lambda > \mu_2$). This may be reasoned as follows : all the existing methods don't properly deal with the complexity associated with the blocking phenomena in the subsequent phases, so look more dependent on the parameters of the initial phase. It is interesting to note that Bell's and S-J's updated upper bounds are same, which are superior to other two upper bounds for all the test cases.

Table II shows five throughput lower bounds : three by the existing methods[6, 7, 8], and two dual bounds obtained from S-J's and VD-L's methods. The dual tightening effect is observed for about the same ratio of the test problems as in the above upper bound case, thereby allowing the same argument too.

Parameters					Exact	Bell		O-P	S-J		VD-L
N_1	N_2	λ	μ_1	μ_2	λ^*	Primal	Dual	Primal	Primal	Dual	Primal
1	1	1.0	1.0	1.0	0.400	0.500	0.500	0.500	0.500	0.500	0.500
2	2	1.0	1.0	1.0	0.578	0.667	0.667	0.667	0.667	0.667	0.667
2	2	1.0	0.5	0.5	0.324	0.429	0.333	0.380	0.429	0.333	0.380
2	2	1.0	0.2	0.2	0.133	0.194	0.133	0.158	0.194	0.133	0.158
2	2	1.0	0.5	0.1	0.097	0.100	0.097	0.100	0.430	0.097	0.100
3	3	1.0	1.0	1.0	0.675	0.750	0.750	0.750	0.750	0.750	0.750
5	3	1.0	1.0	1.0	0.720	0.833	0.750	0.800	0.833	0.750	0.800
5	5	1.0	1.0	1.0	0.778	0.833	0.833	0.833	0.833	0.833	0.833
5	5	1.0	0.1	0.1	0.083	0.100	0.083	0.091	0.100	0.083	0.091
5	5	1.0	0.5	0.1	0.100	0.100	0.100	0.100	0.492	0.100	0.100
5	5	1.0	0.5	0.5	0.416	0.492	0.417	0.450	0.492	0.417	0.450
5	5	1.0	0.5	2.0	0.492	0.492	0.500	0.500	0.492	0.500	0.500
5	5	1.0	2.0	2.0	0.982	0.984	1.000	0.997	0.984	1.667	0.997
5	10	1.0	0.1	0.1	0.091	0.100	0.091	0.094	0.100	0.100	0.094
5	10	1.0	0.5	0.5	0.453	0.492	0.455	0.467	0.492	0.455	0.467
5	10	1.0	2.0	2.0	0.984	0.984	1.000	1.000	0.984	1.818	1.000

Bell : Upper Bounds by Bell[1]

O-P : Upper Bounds by Onvural and Perros[6]

S-J : Upper Bounds by Shathikumar and Jafari[7]

VD-L : Upper Bounds by Van Dijk and Lamond[8]

Table I. Comparison of Throughput Upper Bounds

Parameters					Exact	O-P	S-J		VD-L	
N_1	N_2	λ	μ_1	μ_2	λ^*	Primal	Primal	Dual	Primal	Dual
1	1	1.0	1.0	1.0	0.400	0.333	0.333	0.333	0.333	0.333
2	2	1.0	1.0	1.0	0.578	0.500	0.526	0.526	0.500	0.500
2	2	1.0	0.5	0.5	0.324	0.294	0.307	0.308	0.273	0.300
2	2	1.0	0.2	0.2	0.133	0.128	0.131	0.131	0.107	0.130
2	2	1.0	0.5	0.1	0.097	0.095	0.096	0.096	0.088	0.096
3	3	1.0	1.0	1.0	0.675	0.600	0.634	0.634	0.600	0.600
5	3	1.0	1.0	1.0	0.720	0.600	0.678	0.696	0.652	0.652
5	5	1.0	1.0	1.0	0.778	0.714	0.749	0.749	0.714	0.714
5	5	1.0	0.1	0.1	0.083	0.083	0.083	0.083	0.052	0.083
5	5	1.0	0.5	0.1	0.100	0.100	0.100	0.100	0.090	0.100
5	5	1.0	0.5	0.5	0.416	0.402	0.413	0.414	0.326	0.411
5	5	1.0	0.5	2.0	0.492	0.488	0.492	0.492	0.488	0.397
5	5	1.0	2.0	2.0	0.982	0.950	0.970	0.967	0.968	0.896
5	10	1.0	0.1	0.1	0.091	0.083	0.091	0.091	0.052	0.091
5	10	1.0	0.5	0.5	0.453	0.402	0.451	0.450	0.329	0.448
5	10	1.0	2.0	2.0	0.984	0.950	0.983	0.977	0.983	0.938

O-P : Lower Bounds by Onvural and Perros[6]

S-J : Lower Bounds by Shathikumar and Jafari[7]

VD-L : Lower Bounds by Van Dijk and Lamond[8]

Table II. Comparison of Throughput Lower Bounds

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