Bull. Korean Math. Soc. 27(1990), No. 2, pp. 103-112

ON MATRIX NORMED SPACES

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1. Introduction

Let N and C denote the set of positive real numbers and complex numbers, respectively and let E be a vector space over $\boldsymbol{C}$. Throughout this paper let $M_{m,n}(E)$ denote the vector space of $m\times n$ matrices with entries from E, let $M_{m,n}$ denote the $m\times n$ complex matrices with C^*-norm, and let $\{E_{ij}\}$ denote the standard matrix units for $M_{m,n}$, that is, E_{ij} is 1 in the (i,j)-entry and 0 elsewhere. We set $M_n(E)=M_{n,n}(E)$ and $M_n=M_{n,n}$.

For $x=[x_{ij}]\in M_{k,l}(E)$, $y=[y_{ij}]\in M_{m,n}(E)$, $\alpha=[\alpha_{ij}]\in M_{s,k}$ and $\beta=[\beta_{ij}]\in M_{l,t}$, we write $x\oplus y=\begin{bmatrix} x & 0 \\ 0 & y\end{bmatrix}\in M_{k+m,l+n}(E)$, $\alpha x=[z_{ij}]\in M_{s,l}(E)$, and $x\beta=[w_{ij}]\in M_{k,t}(E)$, where $z_{ij}=\sum_{p=1}^{k}\alpha_{ip}x_{pj}$ and $w_{ij}=\sum_{p=1}^{l}\beta_{pj}x_{ip}$. Here we use the symbol 0 for a rectangular matrix of zero element over E.

If for each $m,n\in\boldsymbol{N}$, there is a norm $\|\cdot\|_{m,n}$ on $M_{m,n}(E)$, the family of the norms $\{\|\cdot\|_{m,n}\}$ is called a matrix norm on E. E is called a space with a matrix norm. We set $\|\cdot\|_n=\|\cdot\|_{n,n}$.

A space E with a matrix norm is called a matrix normed space if for $\alpha\in M_{n,p}$, $x\in M_{p,q}(E)$ and $\beta\in M_{q,m}$, $\|\alpha x\beta\|_{n,m}\leq\|\alpha\|\|x\|_{p,q}\|\beta\|$.

Suppose that E and F are matrix normed spaces and $\phi: E\rightarrow F$ is a linear map. We define the map $\phi_n: M_n(E)\rightarrow M_n(F)$ by $\phi_n([x_{ij}])=[\phi(x_{ij})]$ for $[x_{ij}]\in M_n(E)$. We write $\|\phi\|_{cb}=\sup\{\|\phi_n\|: n\in\boldsymbol{N}\}$. We call ϕ completely bounded if $\|\phi\|_{cb}<\infty$, and completely contractive if $\|\phi\|_{cb}\leq 1$. We call ϕ a complete isometry if for each $n\in\boldsymbol{N}$, $\phi_n: M_n(E)\rightarrow M_n(F)$ is an isometry. Two matrix normed spaces are completely isometrically isomorphic if there is a complete isometry of the first

Received September 25, 1989.
Revised February 13, 1990.
This research was supporeted by KOSEF, 1986-1989.

space onto the second.

A matrix normed space E is an abstract operator space or L^∞-matrix normed space if it satisfies $\|x\oplus y\|_{m+n}=\max\{\|x\|_m, \|y\|_n\}$ and an L^p-matrix normed space or $L^p(1\leq p<\infty)$ if it satisfies $\|x\oplus y\|_{m+n}=(\|x\|_m^p+\|y\|_n^p)^{\frac{1}{p}}$.

In this paper, we study the fundamental properties of matrix normed spaces, the relations between matrix norms and norms, and direct sum of matrix normed spaces.

2.1. Matrix normed spaces

Proposition 2.1. *Let $M_n(E)$ be a normed space for each $n\in \boldsymbol{N}$. Suppose that these norms on $M_n(E)$ satisfy*

(1) *For $x\in M_m(E)$ and $0\in M_n(E)$, we have $\|x\oplus 0\|_{m+n}=\|x\|_m$,*

(2) *For $x\in M_n(E)$, $\alpha, \beta\in M_n$, we have $\|\alpha x\beta\|_n\leq\|\alpha\|\|x\|_n\|\beta\|$.*

For an elemens $x\in M_{m,n}(E)$, we define $\|x\|_{m,n}=\|[x, 0]\|_m$ for $m\geq n$ and $\|x\|_{m,n}=\left\|\begin{bmatrix} x \\ 0\end{bmatrix}\right\|_n$ for $m<n$. Then $(E, \{\|\cdot\|_{m,n}\})$ is a matrix normed space.

Proof. It is trivial to show that $(M_{m,n}(E), \|\cdot\|_{m,n})$ is a normed space. For $\alpha\in M_{m,p}$, $x\in M_{p,q}(E)$, $\beta\in M_{q,n}$, we consider $\alpha\oplus 0$, $\beta\oplus 0\in M_k$ and $x\oplus 0\in M_k(E)$, where $k=\max\{m, n, p, q\}$. Then $[\alpha\oplus 0][x\oplus 0][\beta\oplus 0]=\alpha x\beta\oplus 0\in M_k(E)$. Hence $\|\alpha x\beta\oplus 0\|_k\leq\|\alpha\oplus 0\|\|x\oplus 0\|_k\|\beta\oplus 0\|$. Thus $\|\alpha x\beta\|\leq\|\alpha\|\|x\|_{p,q}\|\beta\|$.

In particular, if $(E, \|\cdot\|_{m,n})$ is a matrix normed space and if $(E, \|\cdot\|'_{m,n})$ is the matrix normed space constructed from $(E, \|\cdot\|_n)$ in Proposition 2.1, then $\|\cdot\|'_{m,n}=\|\cdot\|_{m,n}$. For these reasons we shall often only assign norms to $M_n(E)$ and verify (1) and (2) in Proposition 2.1 to construct a matrix normed space.

Proposition 2.2. *Let $(E, \{\|\cdot\|_{m,n}\})$ be a matrix normed space, $x\in E$ and $\alpha=[\alpha_{ij}]\in M_{m,n}$. Then $\|[\alpha_{ij}x]\|_{m,n}\geq\|\alpha\|\|x\|$. In particular, if E is an abstract operator space, then $\|[\alpha_{ij}x]\|_{m,n}=\|\alpha\|\|x\|$ (cf. [1, Example 2.5]).*

Proof. Since $\alpha^*\alpha\geq 0$, there exists a diagonal matrix $D=\sum_{i=1}^n \lambda_i E_{ii}$

with $\|\alpha\|^2=\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq 0$ and a unitary matrix $U\in M_n$ such that $\alpha^*\alpha=UDU^*$. Hence we have $\|\alpha^*\alpha[x\oplus\cdots\oplus x]\|_n=\|D[x\oplus\cdots\oplus x]\|_n\geq\lambda_1\|x\|$. Therefore $\|\alpha[x\oplus\cdots\oplus x]\|_{m,n}\geq\|\alpha\|\|x\|$. If E is an abstract operator space, we have $\|\alpha\ [x\oplus\cdots\oplus x]\|_{m,n}\leq\|\alpha\|\|x\|$.

COROLLARY 2.3. *Let* $(\boldsymbol{C}, \{\|\cdot\|_n\})$ *be the operator space with the usual matrix norm, let* E *be an abstract operator space, and let* $f:\boldsymbol{C}\to E$ *be a contraction. Then* $\|f\|_{cb}=\|f\|$.

THEOREM 2.4. *Let* $(E, \{\|\cdot\|_{m,n}\})$ *be a matrix normed space,* $\alpha=[\alpha_{ij}]\in M_{m,n}$ *with rank* $\alpha=1$ *and* $x\in E$. *Then* $\|[\alpha_{ij}x]\|_{m,n}=\|\alpha\|\|x\|$.

Proof. Since rank $\alpha=1$, there exist $k_1,\cdots,k_m, a_1,\cdots,a_n\in\boldsymbol{C}$ with $\alpha=[k_1,\cdots,k_m]^t[a_1,\cdots,a_n]$. By elementary calculation, $\|\alpha\|=\|[k_1,\cdots,k_m]\|\|[a_1,\cdots,a_n]\|$. Since $[\alpha_{ij}x]=[k_1,\cdots,k_m]^t[a_1,\cdots,a_n][x\oplus\cdots\oplus x]=[k_1,\cdots,k_m]^t[a_1x,\cdots,a_nx]$, we have $\|[\alpha_{ij}x]\|_{m,n}\leq\|[k_1,\cdots,k_m]\|\|[a_1x,\cdots,a_nx]\|_{1,n}$. Since $[a_1x,\cdots,a_nx]=x[a_1,\cdots,a_n]$, $\|[a_1x,\cdots,a_nx]\|_{1,n}\leq\|x\|\|[a_1,\cdots,a_n]\|$. Hence we have $\|[\alpha_{ij}x]\|_{m,n}=\|\alpha\|\|x\|$ by Proposition 2.2.

REMARK 2.5. Let E be a space with a matrix norm with $\|[x_{ij}]\|_{m,n}=\sum_{i=1}^m\sum_{j=1}^n\|x_{ij}\|$ for each $[x_{ij}]\in M_{m,n}(E)$. Since $\|\sum_{i=1}^m\sum_{j=1}^n E_{ij}x\|=mn\|x\|$, E is not a matrix normed space.

THEOREM 2.6. *Let* E *be a matrix normed space and* $x_{ij}\in M_{k_il_j}(E)$. *Then* $\|[x_{ij}]\|_{s,t}\leq\|[\|x_{ij}\|_{k_il_j}]\|$ *for each* $[x_{ij}]\in M_{s,t}(E)$ *where* $s=k_1+\cdots+k_m$ *and* $t=l_1+\cdots+l_n$ *if and only if* E *is an abstract operator space.*

Proof. ($\Leftarrow$) We may assume that $E\subseteq B(H)$ for some Hilbert space [7, Theorem 1.21]. Note that

$$\left|\left(\begin{bmatrix} x_{11} & \cdots & x_{1n}\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ x_{m1} & \cdots & x_{mn}\end{bmatrix}\begin{bmatrix}\xi_1\\ \cdot\\ \cdot\\ \xi_n\end{bmatrix}\ \middle|\ \begin{bmatrix}\eta_1\\ \cdot\\ \cdot\\ \eta_m\end{bmatrix}\right)\right|$$

$$=\left|\sum_{i=1}^m\sum_{k=1}^n(x_{ik}\xi_k|\eta_i)\right|\leq\sum_{i=1}^m\sum_{k=1}^n\|x_{ik}\|\ \|\xi_k\|\ \|\eta_i\|$$

$$=\left(\begin{bmatrix}\|x_{11}\| & \cdots & \|x_{1n}\|\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ \|x_{m1}\| & \cdots & \|x_{mn}\|\end{bmatrix}\begin{bmatrix}\|\xi_1\|\\ \cdot\\ \cdot\\ \|\xi_n\|\end{bmatrix}\ \middle|\ \begin{bmatrix}\|\eta_1\|\\ \cdot\\ \cdot\\ \|\eta_n\|\end{bmatrix}\right),$$

where $\xi_i \in H^{k_i}$ and $\eta_i \in H^{l_i}$. Hence $\|[x_{ij}]\|_{s,t} \leq \|[\|x_{ij}\|]\|$.

($\Rightarrow$) Suppose E is not an abstract operator space. Then there are $x \in M_m(E)$ and $y \in M_n(E)$ such that $\|x \oplus y\|_{m+n} > \max\{\|x\|, \|y\|\}$. But $\|x \oplus y\|_{m+n} = \left\|\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right\| \leq \left\|\begin{bmatrix} \|x\| & 0 \\ 0 & \|y\| \end{bmatrix}\right\| = \max\{\|x\|, \|y\|\}$, which contradicts to the choices of x and y. Hence $\|[x_{ij}]\|_{s,t} \leq \|[\|x_{ij}\|]\|$ implies that E is an abstract operator space.

3. Constructions of matrix norms

For each $p (1 \leq p \leq \infty)$, we define $\|\cdot\|_{p,n}$ on M_n by $\|x\|_{p,n} = (tr(x^p))^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|x\|_{\infty,n} = \|x\|$, where tr is the canonical trace on M_n, and $\|\cdot\|$ is the usual operator norm on M_n.

Let $(E, \|\cdot\|)$ be a normed space. Then $\Omega = (E^*)_1$, the closed unit ball of E^*, is a compact Hausdorff space with respect to w^*-topology. Then $x(f) = (x, f)$ gives an isometric injection $E \to C(\Omega)$, where $C(\Omega)$ is the C^*-algebra of all complex-valued continuous functions on Ω with the sup-norm.

For a $x = [x_{ij}] \in M_n(E)$, define ${}_p\|x\|_n = \sup\{\|[f(x_{ij})]\|_{p,n} : f \in \Omega, \|f\| = 1\}$. Since $(\boldsymbol{C}, \|\cdot\|_{p,n})$ is L^p, $(E, \{{}_p\|\cdot\|_n\})$ is an L^p matrix normed space.

Proposition 3.1. *Let E be a matrix normed space, let Ω be a locally compact Hausdorff space, and let $f: E \to C_0(\Omega)$ a bounded linear map. Then $\|f\|_{cb} = \|f\|$.*

Proof. The same as the proof of [5, Proposition 3.7 and Theorem 3.8].

Proposition 3.2. *Let A and B be C^*-algebras. If $\phi: A \to B$ is positive and completely contractive, then ϕ is completely positive.*

Proof. If A and B are unital, then it follows from [5, Proposition 3.4]. For non-unital A and B, let $A \oplus \boldsymbol{C}$ and $B \oplus \boldsymbol{C}$ be the unital C^*-algebra obtained from A and B by the adjunction of an identity, respectively. If $a + \lambda I$ is positive, then there is an element $b + \mu \in A \oplus \boldsymbol{C}$ with $a + \lambda I = (b + \mu I)^*(b + \mu I)$, and $\lambda = |\mu|^2 \geq 0$. Define $\tilde{\phi}: A \oplus \boldsymbol{C} \to B \oplus \boldsymbol{C}$ by $\tilde{\phi}(a + \lambda I) = \phi(a) + \lambda I$. If $a + \lambda I \geq 0$, then $a^- \leq \lambda I$. Since ϕ is contractive and positive, $\phi(a^-)\leq\lambda I$. Hence $-\phi(a)=\phi(a^-)-\phi(a^+)\leq\lambda I$, so $\tilde{\phi}$ is positive. By [5, Proposition 3.4], $\tilde{\phi}$ is completely positive.

Let $\theta(n)$ be the transpose map in M_n. Then the norms of the multiplicity maps $\theta(n)_k$ are $\|\theta(n)_k\|=k$ if $k\leq n$ and $\|\theta(n)_k\|=n$ if $k>n$ [8, Theorem 1.2].

LEMMA 3.3. *Let A be a C^*-algebra and $\theta(2, A)$ the transpose map in $M_2(A)$. Then A is non-commutative if and only if $\|\theta(2, A)\|=2$.*

Proof. ($\Leftarrow$) Suppose that A is commutative. Then $\|\theta(2, A)\|=1$.
($\Rightarrow$) Since A is non-commutative, there is an irreducible representation $\{\phi, H\}$ with $\dim H\geq 2$. Since $\phi_2\circ tr(2, A)=tr(2, B(H))\circ\phi_2$ and $\|\theta(2, M_n)\|=2$ for $n\geq 2$, $\|\theta(2, A)\|=2$.

Let $(E, \{\|\cdot\|_n\})$ be a matrix normed space. Define new matrix norms on $M_n(E)$ by ${}_\infty\|[x_{ij}]\|_n=\sup\{\|[f(x_{ij})]\| : f\in(E^*)_1\}$ and ${}^\infty\|[x_{ij}]\|_n=\sup\{\|[\phi(x_{ij})]\|_n\}$, where the supremum is taken over all Hilbert spaces H and all contractive linear maps ϕ from E to $B(H)$. Since $f\in(E^*)_1$ is completely contractive, ${}_\infty\|[x_{ij}]\|_n\leq\|[x_{ij}]\|_n$. Hence the matrix norm $\{{}_\infty\|\cdot\|_n\}$ is the minimum of all possible matrix norms on E. Clearly $(E, \{{}^\infty\|\cdot\|_n\})$ and $(E, \{{}_\infty\|\cdot\|_n\})$ are abstract operator spaces, and $\{{}^\infty\|\cdot\|_n\}$ is the maximum of all possible norms which make it an abstract operator space. Since for any matrix norm $\{\|\cdot\|_n\}$ $\|[x_{ij}]\|_n\leq\sum_{ij=1}^n\|x_{ij}\|$, $\|[x_{ij}]\|_n^{\max}=\sup\{\|[x_{ij}]\|_n : E$ is a matrix normed space with $\{\|\cdot\|_n\}\}$ is the maximum matrix norm which makes E a matrix normed space.

THEOREM 3.4. *Let A be a commutative C^*-algebra. Then the usual matrix norm on A is the minimum matrix norm making A a matrix normed space, but for any non-commutative C^*-algebra A the usual matrix norm is not the minimum matrix norm making A a matrix normed space.*

Proof. First suppose that A is commutative. Then we may assume that $A=C_0(\Omega)$ for a locally compact Hausdorff space Ω. Put $E=A$ with a matrix norm which makes A a matrix normed space and $C_0(\Omega)=A$ with the usual norm. By Proposition 3.1, the usual matrix norm on A is the minimum matrix norm on A.

Now suppose that A is not commutative. Then the transpose map $\theta(2, A) : M_2(A) \to M_2(A)$ is not contractive by Lemma 3. 3. By elementary calculation, $(A, \{_\theta\| \cdot \|_n\})$ is a matrix normed space, where $_\theta\|x\|_n = \|{}^t x\|_n$. It is trivial to show that if $(E, \{\| \cdot \|_n\})$ is an abstract operator space then $(E, \{_\theta\| \cdot \|_n\})$ is an abstract operator space. Hence the usual matrix norm on A is not the minimum matrix norm on A.

THEOREM 3. 5. *Let A be a C^*-algebra. Then A has only one abstract operator space structure if and only if A is at most two dimensional.*

Proof. ($\Leftarrow$) By [9, Proposition 3. 1 (b)], $_\infty\|[x_{ij}]\|_n = {}^\infty\|[x_{ij}]\|_n$ for each $[x_{ij}] \in M_n(A)$. Hence A has only one abstract operator space structure.

($\Rightarrow$) Suppose that A is at least three dimensional. Then by [9, Proposition 3. 1 (b)], $_\infty\|[x_{ij}]\|_n \neq {}^\infty\|[x_{ij}]\|_n$. Hence A has at least two abstract operator spaces structure.

Let $l_n{}^p = \{[a_1, \cdots, a_n]^t : a_1, \cdots, a_n \in \boldsymbol{C}\}$ be a Banach space with a norm $_p\|[a_1, \cdots, a_n]^t\|_n{}^p = |a_1|^p + \cdots + |a_n|^p$. Considering $\alpha \in M_n$ as a linear transformation from $l_n{}^p$ to $l_n{}^q$, we define a new norm $_{p,q}\| \cdot \|_n$ on M_n.

PROPOSITION 3. 6. *$(\boldsymbol{C}, \{_{p,q}\| \cdot \|_n\})$ is a matrix normed space if and only if $p = q = 2$.*

Proof. ($\Leftarrow$) Clear.

($\Rightarrow$) By Theorem 2. 4, $_{p,q}\left\|\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\|_2 = 2$. $_{p,p}\left\|\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right\|_2 = \sqrt{2}$. By elementary calculation, $_{p,q}\left\|\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right\|_2 = 2^{1-\frac{1}{p}}$ and $_{p,q}\left\|\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right\|_2 = 2^{\frac{1}{q}}$. Thus $p = q = 2$.

COROLLARY 3. 7. *The norm $\| \ \|_{p,n}$ on M_n defined by $\|\alpha\|_{p,n} = (tr(|x|^p))^{\frac{1}{p}}$ is an operator norm from $l_n{}^p$ to $l_n{}^q$ for $1 \leq p, q \leq \infty$ if and only if $(p, q) = (2, 2)$.*

PROPOSITION 3. 8. *Let E be a normed space. Define a norm $\| \cdot \|_n$ on $M_n(E)$ by $\|[x_{ij}]\|_n = \|[\|x_{ij}\|]\|$. Then $(E, \{\| \cdot \|_n\})$ is not a matrix normed space.*

Proof. Suppose that E is a matrix normed space. Then for $x \in M_m(E)$ and $y \in M_n(E)$, we have $\|x \oplus y\|_{m+n} = \max\{\|x\|_m, \|y\|_n\}$. Let $u = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then $u^*u = I_2$, so $\|u[x \oplus x]\|_2 = \|x \oplus x\|_2 = \|x\|$. But $\frac{1}{\sqrt{2}}\left\|\begin{bmatrix} x & x \\ x & x \end{bmatrix}\right\|_2 = \frac{1}{\sqrt{2}}\left\|\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\| \|x\| = \sqrt{2}\|x\|$ by Theorem 2.5, a contradiction.

4. Direct sum of matrix normed spaces

Definition 4.1. Let $(E, \{\|\cdot\|_n\})$ and $(F, \{\|\cdot\|_n\})$ be matrix normed spaces. Put $E \oplus_p F = \{x \oplus_p y : x \in E, y \in F\}$ with a norm $\|x \oplus_p y\| = (\|x\|^p + \|y\|^p)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|x \oplus_p y\| = \max\{\|x\|, \|y\|\}$ for $p = \infty$. Identifying $M_n(E \oplus_p F)$ with $M_n(E) \oplus_p M_n(F)$ via $[x_{ij} \oplus_p y_{ij}] = [[x_{ij}] \oplus_p [y_{ij}]]$, $(E \oplus_p F, \{\|\cdot\|_n\})$ becomes a space with a matrix norm. We call $E \oplus_p F$ the *p-direct sum* of E and F.

Theorem 4.2. *$(E \oplus_p F, \{\|\cdot\|_n\})$ is a matrix normed space. Furthermore if E and F are L^p, then $E \oplus_p F$ is L^p.*

Proof. Since $[x_{ij} \oplus_p y_{ij}] \oplus 0 = [[x_{ij}] \oplus 0] \oplus_p [[y_{ij}] \oplus 0]$ for $[x_{ij} \oplus_p y_{ij}] \oplus 0 \in M_{m+n}(E \oplus_p F)$, we have $\|[x_{ij} \oplus_p y_{ij}] \oplus 0\|_{m+n} = \|[x_{ij}] \oplus_p [y_{ij}]\|_m$. For $\alpha = [\alpha_{ij}]$, $\beta = [\beta_{ij}] \in M_n$, $x = [x_{ij}] \in M_n(E)$ and $y = [y_{ij}] \in M_n(F)$, we have $\|\alpha[x_{ij} \oplus_p y_{ij}]\beta\|_n = \|\alpha x \beta \oplus_p \alpha y \beta\|_n \leq \|\alpha\| \|\beta\| \|x \oplus_p y\|_n$. Therefore $E \oplus_p F$ is a matrix normed space.

Let E and F be L^p for $1 \leq p < \infty$, $[x_{ij} \oplus_p y_{ij}] \in M_m(E \oplus_p F)$ and $[z_{kl} \oplus_p w_{kl}] \in M_n(E \oplus_p F)$. Then $\|[x_{ij} \oplus_p y_{ij}] \oplus [z_{kl} \oplus_p w_{kl}]\|_{m+n} = (\|[x_{ij}] \oplus [z_{kl}]\|^p + \|[y_{ij}] \oplus [w_{kl}]\|^p)^{\frac{1}{p}} = (\|[x_{ij} \oplus_p y_{ij}]\|^p + \|[z_{kl} \oplus_p w_{kl}]\|^p)^{\frac{1}{p}}$. Hence $E \oplus_p F$ is L^p. In case that E and F are abstract operator spaces and $p = \infty$, similarly it can be shown that $E \oplus_\infty F$ is an abstract operator space.

Proposition 4.3. *Let E, F, and G be non-zero matrix normed spaces. Then the following are equivalent:*

(1) *$E \oplus_p F$ and $E \oplus_q F$ are completely isometrically isomorphic.*

(2) *$E \oplus_p F$ and $F \oplus_q E$ are completely isometrically isomorphic.*

(3) *$(E \oplus_p F) \oplus_q G$ and $E \oplus_p (F \oplus_q G)$ are completely isometrically isomorphic.*

(4) $p=q$.

For a matrix normed space E, we define the left dual of E, E_l^* to be the dual of E together with the norms on $M_n(E^*)$ obtained by identifying $M_n(E^*)$ with $B(M_{n,1}(E), M_{n,1})$. It is easy to check that E is a matrix normed space with this matrix norm.

EXAMPLE 4.4. Let $E=\left\{\begin{bmatrix} x & y \\ 0 & 0\end{bmatrix} : x, y\in \boldsymbol{C}\right\}$ with the usual operator norm, F be the left dual of E. Then by elementary calculation, we show that F is not decomposed into p-direct sum.

Let E be a matrix normed space. There are two natural ways to identify $M_n(E^*)$ with $M_n(E)^*$. The first way is defined by$([x_{ij}], [f_{ij}])=\sum_{i,j=1}^{n}(x_{ij}, f_{ji})$. We denote this dual space ${}_1E^*$. Another way is defined by $([x_{ij}], [f_{ij}])=\sum_{i,j=1}^{n}(x_{ij}, f_{ji})$. We denote this dual space ${}_2E^*$. We know that these dual spaces are matrix normed spaces [7, Proposition 1.1.6 and Appendix].

THEOREM 4.5. *Let E and F be matrix normed spaces. Then ${}_\alpha(E\oplus_p F)^*$ is completely isometrically isomorphic to ${}_\alpha E^*\oplus_{q\alpha} F^*$, where $\frac{1}{p}+\frac{1}{q}=1$, $1\le p, q$ and $\alpha=1$ or 2.*

Proof. Define $\phi : {}_\alpha E^*\oplus_{q\alpha}F^*\rightarrow {}_\alpha(E\oplus_q F)^*$ by $(\phi(f\oplus_q g))(x\oplus_p y)=f(x)+g(y)$ for $f\in {}_\alpha E^*$, $g\in {}_\alpha F^*$, $x\in E$ and $y\in F$.

Case 1. $1\le p, q<\infty$: Note that for $[f_{ij}\oplus_q g_{ij}]\in M_n({}_\alpha E^*\oplus_{q\alpha}F^*)$,

$$
\begin{aligned}
&{}_\alpha\|\phi_n([f_{ij}\oplus_q g_{ij}])\|_n\\
&=\sup\{|\phi_n([f_{ij}\oplus_q g_{ij}])([x_{ij}\oplus_p y_{ij}])| : \|[x_{ij}\oplus_p y_{ij}]\|_n=1\}\\
&=\sup\{|([x_{ij}], [f_{ij}])+([y_{ij}], [g_{ij}])| : \|[x_{ij}]\|^p{}_n+\|[y_{ij}]\|^p{}_n=1\}\\
&=\sup\{{}_\alpha\|[f_{ij}]\|_n\|[x_{ij}]\|_n+{}_\alpha\|[g_{ij}]\|_n\|[y_{ij}]\|_n : \|[x_{ij}]\|_n{}^p+\|[y_{ij}]\|_n{}^p=1\}\\
&=({}_\alpha\|[f_{ij}]\|_n{}^q+{}_\alpha\|[g_{ij}]\|_n{}^q)^{\frac{1}{q}}.
\end{aligned}
$$

Hence ϕ is a complete isometry.

Case 2. $p=1$ or $p=\infty$: By the same way, it holds.

In the classical functional analysis, we know that if $(E, \|\cdot\|)$ is a normed space and E_0 is a closed subspace of E then the quotient space

$\frac{E}{E_0}$ with the quotient norm.

Now, suppose that $(E, \{\|\cdot\|_n\})$ is a matrix normed space and that E_0 is a subspace of E which is closed under the norm $\|\cdot\|_1$. Then each $M_n(E_0)$ is closed in $M_n(E)$ under the norm $\|\cdot\|_n$. Identifying $M_n\left(\frac{E}{E_0}\right)$ with $\frac{M_n(E)}{M_n(E_0)}$, we may let $M_n\left(\frac{E}{E_0}\right)$ have the corresponding quotient norm $\|\cdot\|_n$. It is known that $\frac{E}{E_0}$ is a matrix normed space [7, Theorem 1.1.8].

Proposition 4.6. *Let E and F be matrix normed spaces, and let E_0 and F_0 be closed subspaces, respectively. Then $\frac{E\oplus_p F}{E_0\oplus_p F_0}$ is completely isometrically isomorphic to $\frac{E}{E_0}\oplus_p\frac{F}{F_0}$.*

Proof. Define $\phi : \frac{E\oplus_p F}{E_0\oplus_p F_0}\to\frac{E}{E_0}\oplus_p\frac{F}{F_0}$ by $\phi(\overline{f\oplus_p g})=\bar{f}\oplus_p\bar{g}$. Then clearly ϕ is well defined.

Note that $\phi_n(\overline{[x_{ij}\oplus_p g_{ij}]})=[\bar{x}_{ij}]\oplus_p[\bar{y}_{ij}]$. Hence $\|\overline{[x_{ij}\oplus_p y_{ij}]}\|_n=(\|\overline{[x_{ij}]}\|_n^p+\|\overline{[y_{ij}]}\|_n^p)^{\frac{1}{p}}$. Therefore ϕ is a complete isometry.

Remark 4.7. $\left(\frac{(E\oplus_p F)}{(E_0\oplus_p F_0)}\right)^*$ is completely isometrically isomorphic to $\left(\frac{E}{E_0}\oplus_p\frac{F}{F_0}\right)^*$ and $\left(\frac{E}{E_0}\oplus_p\frac{F}{F_0}\right)^*$ is completely isometrically isomorphic to $\left(\frac{E}{E_0}\right)^*\oplus_q\left(\frac{F}{F_0}\right)^*$, where $\frac{1}{p}+\frac{1}{q}=1$. Hence $\left(\frac{(E\oplus_p F)}{(E_0\oplus_p F_0)}\right)^*$ is completely isometrically isomorphic to $\left(\frac{E}{E_0}\right)^*\oplus_q\left(\frac{F}{F_0}\right)^*$.

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