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Nearly Kaehlerian manifolds with vanishing conformal curvature tensor field

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1. Introduction

Let (M,F,g) be a Kaehlerian manifold of real dimension 2n with almost complex structure F and Kaehlerian metric g.

We shall be in C^∞-category. Latin indicies run from 1 to 2n. The Einstein summation convention will be used.

We denote by g_{ji}, ∇_i, $K_{sji}{}^t$, K_{ji}, r and F_j^i local components of g, the operator of covariant differentiation with respect to the Riemannian conncetion, the curvature tensor, the Ricci tensor, the scalar curvature and F of M respectively, which satisfy ([4])

$$(1.1)\qquad F_j^t F_t^i = -\delta_j^i,\ F_j^s F_i^t g_{st} = g_{ji},\ F_{ji} = -F_{ij},$$
$$\nabla_k F_j^h = 0,\ F_j^s F_i^t K_{st} = K_{ji}.$$

Put $F_{ji} = F_j^t g_{ti}$, then we have

$$(1.2)\qquad \nabla_k F_{ji} = 0$$

In 1989, H. Kitahara, K. Matsuo and J. S. Pak ([3]) defined a new tensor field on a hermitian maninfold which is conformally invariant and studied some properties of this new tensor field. They called this new tensor field *confomal curvature tensor field*. In particular, on a 2n dimensional Kaehlerian manifold the conformal curvature tensor B_0 is given by

$$(1.3)\qquad B_{0sjit} = K_{sjit} + \frac{1}{2n}(g_{ji}K_{st} - g_{si}K_{jt} + K_{ji}g_{st}$$
$$- K_{si}g_{jt} - F_{ji}S_{st} + F_{si}S_{jt} - S_{ji}F_{st}$$
$$+ S_{si}F_{jt} + 2F_{it}S_{sj} + 2S_{it}F_{sj})$$
$$+ \frac{(n+2)r}{4n^2(n+1)}(F_{ji}F_{st} - F_{si}F_{jt} - 2F_{it}F_{sj})$$
$$- \frac{(3n+2)r}{4n^2(n+1)}(g_{ji}g_{st} - g_{si}g_{jt}),$$

where $S_{ji} = F_j^t K_{ti}$.

It is well known that a Kaehlerian manifold with vanishing Bochner curvature tensor is a complex analogue to a conformally flat Riemannian manifold and that the Bochner curvature tensor has properties quite similar to those of Weyl conformal curvature tensor (cf. [6]).

An almost hermitian manifold M is called *a nearly Kaehlerian manifold* if it holds that ([1])

$$(1.4)\qquad \nabla_k F_{ji} + \nabla_j F_{ki} = 0.$$

Every Kaehlerian manifold is a nearly Kaehlerian manifold, but the converse is false in general. However, A. Gray ([2]) proved the following Theorem A.

Theorem A. *Any nearly Kaehlerian manifold of dimension 2n (n≦2) is Kaehlerian manifold.*

In this paper, we shall study nearly Kaehlerian manifold with vanishing conformal curvature tensor field and prove the following theorem generalized Theorem A :

Theorem B. *Any nearly Kaehlerian manifold of dimension 2n with vanishing conformal curvature tensor field is Kaehlerian manifold.*

2. Proof of Theorem B

Let M be a nearly Kaehlerian manifold of real dimension 2n. Then we have

(2.1) $$F_{ji}F^{ji} = \text{constant},$$

$$\triangle(F_{ji}F^{ji}) = g^{st}\nabla_s\nabla_t(F_{ji}F^{ji}) = 0,$$

where $\triangle$ is the Laplacian operator.

By [5], we hvae

(2.2) $$\frac{1}{2}g^{st}\nabla_s\nabla_t(F_{ji}F^{ji}) = (g^{st}\nabla_s\nabla_t F_{ji})F^{ji} + \| \nabla_k F_{ji} \|^2,$$

that is,

(2.3) $$g^{st}(\nabla_s\nabla_t F_{ji})F^{ji} + \| \nabla_k F_{ji} \|^2 = 0.$$

By virtue of the Ricci formular for F_{ji} and *(1.4)*, we have

$$(2.4) \qquad g^{st}(\nabla_s \nabla_t F_{ji}) F^{ji} = K_{ja} F_i^{\ a} F^{ji} + K_{sjit} F^{st} F^{ji}.$$

Also, we have

$$(2.5) \qquad K_{ja} F_i^{\ a} F^{ji} = -r, g_{ji} F^{ji} = 0 \text{ and } F_{si} F^{st} = -\delta_i^{\ t}.$$

Thus, under the hypothesis that the conformal curvature tensor field vanishies, we have by (1.3) and (2.4)

$$(2.6) \qquad g^{st}(\nabla_s \nabla_t F_{ji}) F^{ji} = -r - \frac{1}{2n}(-4nr)$$
$$- \frac{(n+2)r}{4n^2(n+1)}(4n^2+2n) + \frac{(3n+2)r}{4n^2(n+1)}(2n) = 0.$$

Thus, we have by (2.3) and (2.6), $\nabla_k F_{ji} = 0$.
Hence Theoren B has been proved.

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