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Spaces of the type Q_w and the Laplace transformation

Young Sik Park

0. Introduction

This article is devoted to the study of the Banach spaces of analytic functions of the type Q_w and the type Q'_w dual which are closely related to the Laplace transformation.

As first, we define spaces of the type Q_w and examine connection between spaces of the type Q_s and Q_w. Especially, Laplace tranforms and inverse Laplace transforms on the spaces of the type Q_w and that of the type Q'_w are studied. In the case of convex compact subsets K, K' in R^n are $\mathscr{L}$-admissible pairs, we considered conditions of existence of a spectral function, and related theorems.

1. The space $Q_w(T(K); K')$ and spaces of the type Q_w.

With a view to constructing the Laplace transformation of analytic functions, we now introduce test-function spaces consisting of functions that are continuous in horizontal bands, holomorphic in the interior of the horizontal bands and of exponentials.

We denote by $Q_w(T(K); K')$ the Banach space consisting of all

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functions ϕ continuous in $T(K)$ and holomorphic in the interior $T(K)$ of $T(K)$ for which

$$\|\phi\|_{K,K'_w}=sup\{e^{W_{K'}(x)}|\phi(z)| : z=x+iy \in T(K)\}<\infty \tag{1.1}$$

where $W_{K'}(x)=\inf\{x\eta : \eta \in K'\}$, and having the topology defined by the norm $\|\cdot\|_{K,K'_w}$(we recall yet once more that K and K' are convex compact subsets in R^n with nonempty interior).

Since $Q_w(T(K);K')\subset Q_w(T(L);L')$ if $L\subset K$, $K'\subset L'$,

the natural embedding mapping

$$i^{KL}_{K'L'} : Q_w(T(K);K')\to Q_w(T(L);L') \tag{1.2}$$

is defined (we emphasize that here $K'\subset L'$, and not $K'\supset L'$, in contrast to the space $Q_s(T(K);K')$).

The mapping $i^{KL}_{K'L'}$ is compact for all $L\Subset K$, $K'\Subset L'$.

The proof is the same as in the case of spaces of the type Q_s.

We shall also assume everywhere that K,K',L and L' are convex compact sets with nonempty interior and U,U' are convex open sets in R^n.

We denote by

$$\vec{Q}_w(T(K);U')=\lim_{\substack{\longrightarrow \\ K\Subset L,U'\supset L'}} ind\ Q_w(T(L);L') \tag{1.3}$$

The inductive limit of the Banach spaces $Q_w(T(L);L')$ is taken over all convex, compact sets $L \supset K$, $L' \Subset U'$.

Since the mappings $i^{LM}_{L'M'}: Q_w(T(L);L') \to Q_w(T(M);M')$ $(M \Subset L$, $L' \Subset M')$ are compact, $\overrightarrow{Q}_w(T(K);U')$ is a space of the type (DFS) and its dual

$$\overrightarrow{Q}_w'(T(K);U') = \lim_{\substack{K \Subset L,\ U' \supset L'}} \mathrm{proj}\ Q_w'(T(L);L') \tag{1.4}$$

is a space of type (FS). Further, we denote by, for an open convex set U and a convex compact set K' of R^n with nonempty interior,

$$\overleftarrow{Q}_w(T(U);K') = \lim_{\substack{L \Subset U,\ L' \supset K'}} \mathrm{proj}\ Q_w(T(L);L') \tag{1.5}$$

the projective limit of the Banach spaces $Q_w(T(L);L')$.
The space $\overleftarrow{Q}_w(T(U);K')$ is a space of the type (FS), and its dual

$$\overleftarrow{Q}_w'(T(U);K') = \lim_{\substack{L \Subset U,\ L' \supset K'}} \mathrm{ind}\ Q_w'(T(L);L') \tag{1.6}$$

is a space of the type (DFS).

We have defined the followings:

$$\overrightarrow{Q}_w(T(K);R^n) = \lim_{K' \Subset R^n} \mathrm{ind}\ Q_w(T(K);K') \tag{1.7}$$

$$\overleftarrow{Q}_w(T(K);(0)) = \lim_{\{0\} \Subset K'} \operatorname{proj} Q_w(T(K);K'), \tag{1.8}$$

$$\overrightarrow{Q}_w(T(0);R^n) = \lim_{\{0\} \Subset K,\ R^n \supset K'} \operatorname{ind} Q_w(T(K);K') \equiv \overrightarrow{Q}_w(R^n) \tag{1.9}$$

$$Q_w(T(0);(0)) = \lim_{K \supset \{0\}} \operatorname{ind}[\lim_{\{0\} \Subset K'} \operatorname{proj} Q_w(T(K);K')] \tag{1.10}$$

We remark that the space $\overrightarrow{Q}_w(R^n)$ is properly included in the space $\overrightarrow{Q}_w(R^n)$. The space $Q_w(T(0);(0))$ is also

$$\lim_{\{0\} \Subset K'} \operatorname{proj}[\lim_{K \supset \{0\}} \operatorname{ind} Q_w(T(K);K')]$$

and hence $Q_w(T(0);(0)) = \lim\limits_{K \supset \{0\}} \operatorname{ind} \overleftarrow{Q}_w(T(K);(0))$

$= \lim\limits_{\{0\} \Subset K'} \operatorname{proj} \overrightarrow{Q}_w(T(0);K')$.

The spaces $Q_w(T(K);K')$ and the spaces obtained from them by means of the inductive and projective limits will be called here spaces of the type Q_w; the dual, of the type Q'_w.

2. The connection between spaces of the type Q_s and Q_w.

By definitions we have

$$Q_s(T(K);-K') \subset Q_w(T(K);K') \quad \text{and} \quad \|\phi\|_{K,K'_w} \leq \|\phi\|_{K,-K'} \tag{2.1}$$

for $\phi \in Q_s(T(K); -K')$. For in accordance with property g) of the nega-support function [11] $W_{K'} \leq -W_{-K'}$, whence $\|\phi\|_{K,K'_w} \leq \|\phi\|_{K,-K'}$

For the decreasing sequence of positive numbers $\varepsilon_j > 0 (j=0,1,2,\ldots)$ such that $\lim_{j\to\infty} \varepsilon_j = 0$, $\varepsilon_0 = \varepsilon$, we have the following relations:

$$Q_s(T(K); [-\varepsilon,\varepsilon]^n \subset Q_s(T(K); [-\varepsilon_1,\varepsilon_1]^n) \subset \ldots \subset Q_s(T(K); [-\varepsilon_j,\varepsilon_j]^n) \subset \ldots \subset$$
$$\vec{Q}_s(T(K); (0)) \subset \overleftarrow{Q}_w(T(K); (0)) \subset \ldots \subset$$
$$Q_w(T(K); [-\varepsilon_j,\varepsilon_j]^n) \subset \ldots \subset Q_w(T(K); [-\varepsilon,\varepsilon]^n). \tag{2.2}$$

Under the condition $\{0\} \Subset K'$, we have the followings:

$$\overleftarrow{Q}_s(C^n) \subset Q_s(T(K); R^n) \subset Q_s(T(K); K') \subset \vec{Q}_s(T(K); (0)) \subset \overleftarrow{Q}_w(T(K); (0))$$
$$\subset Q_w(T(K); K') \subset \overrightarrow{Q}_w(T(K); R^n) \subset \overrightarrow{Q}_w(R^n) \tag{2.3}$$

Proposition 2.1. Then space $\overleftarrow{Q}_s(C^n)$ is dense in $Q_w(T(K); K')$ in the topology of $Q_w(T(L); L')$ if $L \Subset K$, $L' \supset K'$

For first, as in the case of the space $Q_s(T(K); K')$, we can show that $Q_s(T(K); R^n)$ is dense in $Q_w(T(K); K')$ in the topology of $Q_w(T(L); L')$ for $L' \supset K' \supset \{0\}$. Further, as we have shown [11] Proposition 4.4, $\overleftarrow{Q}_s(C^n)$ is dense in $Q_s(T(K); R^n)$ in the toplogy of $Q_s(T(L); -L') \subset Q_w(T(L); L')$. Therefore, for every function $\phi \varepsilon Q_s(T(K); R^n)$ there exists a sequence $\{\phi_k\} \subset \overleftarrow{Q}_s(C^n)$ such that $\|\phi-\phi_k\|_{L,-L'} \to 0$ as $k\to\infty$.

But then $\|\phi-\phi_k\|_{L,L'_w} (\leq \|\phi-\phi_k\|_{L,-L'}) \to 0$ as $k\to\infty$. i.e., $\overleftarrow{Q}_s(C^n)$ is dense in $Q_s(T(K); R^n)$ in the topology of $Q_w(T(L); L')$ as well.

Thus, $\overleftarrow{Q}_s(C^n)$ is dense in $Q_s(T(K);R^n)$, and $Q_s(T(K);R^n)$ is dense in $Q_w(T(K);K')$, whence the assertion.

We have the followings:

$$\overleftarrow{Q}_s(C^n) \subset \overrightarrow{Q}_w(T(K);U') \subset \overrightarrow{Q}_w(T(K);R^n), \tag{2.4}$$

$$\overleftarrow{Q}_s(C^n) \subset \overleftarrow{Q}_s(T(R^n);(U')) \subset \overleftarrow{Q}_w(T(R^n);(U')) \subset \overleftarrow{Q}_w(T(R^n);K') \subset \overleftarrow{Q}_w(T(U);K'), \text{ if } \{0\} \Subset K', \text{ and } U' \supset \{0\}. \tag{2.5}$$

The space $\overleftarrow{Q}_s(C^n)$ is dense in $\overleftarrow{Q}_w(T(U);K')$ and in $\overrightarrow{Q}_w(T(K);U')$. This follows directly from the previous assertion.

3. Construction of the Laplace transformation.

We now turn to construction of the Laplace transformation. It is readily verified that $e^{iz}\in Q_w(T(K);K')$ for all $z\in T(K')$, and $\|e^{iz}\|_{K,K'_w}\leq e^{-w_K(x)}$ $(z=x+iy)$. Moreover, differentiation of the exponential e^{iz} with respect to the parameter z is continuous in $Q_w(T(K);K')$. More precisely, suppose $z\in T(K')$ then

$$\left\|\frac{e^{i(z+\Delta_j z)}-e^{iz}}{\Delta z_j}-\frac{\partial}{\partial z_j}e^{iz}\right\|_{K,K_w}\to 0 \text{ as } \Delta z_j\to 0,$$

$j=1,2,\ldots n$, where $\Delta_j z=(0,\ldots,0,\ \Delta z_j,0,\ldots,0)\in C^n$, and Δz_j is the j-th component.

We denote by $Q'_w(T(K);K')$ the space dual to $Q_w(T(K);K')$ and define the Laplace tranform $\mathscr{L}[g]$ of an analytic functional g $\varepsilon Q'_w(T(K);K')$ by the equation

$$\mathscr{L}[g](z)=(g,e^{iz}),$$

i.e., we define the Laplace transform of a functional by its values on exponentials. By virtue of what we have said above, the function $f(z)=\mathscr{L}[g](z)$ is defined and holomorphic in $T(\dot{K}')$ and satisfies the estimate

$$|f(z)| \leq \|g\| \, \|e^{iz}\|_{K \cdot K'_w} \leq \|g\| \, e^{-W_K(x)},$$

where $z=x+iy$ and $\|g\|$ is the norm of the functional g.

Theorem 3.1. Suppose that $L,K,L',\&K'$ are convex compact sets in R^n with nonempty interior and that $K \Subset L$, $L' \Subset K'$. Then the Laplace transformation $\mathscr{L}$ is a continuous linear mapping of

$$Q'_w(T(K)\,;\,K')\to Q_w(T(L')\,;\,L)$$

and $\|\mathscr{L}[g]\|_{L'L_w}=\|f\|_{L'L_w}\leq\|g\|$.

This permits us to transfer the Laplace transformation $\mathscr{L}$ to the projective and inductive limits of the spaces $Q'_w(T(K)\,;\,K')$.

Theorem 3.2. The Laplace transformation that associates every analytic functional $g \in \overrightarrow{Q'}_w(T(K)\,;\,U)$ with the function $f(z)=(g,e^{iz})$, $z \in T(U)$, defines a continuous linear mapping

$$\mathscr{L} : \overrightarrow{Q'_w}(T(K)\,;\,U)\to\overleftarrow{Q}_w(T(U)\,;\,K).$$

4. Inversion of the Laplace transformation on the spaces $Q_w(T(K');K)$.

Definition 4.1. Suppose $f \in Q_w(T(K');K)$. We shall call every analytic functional $g \in Q'_w(T(L);L')$, where $L \supset K$, $L' \Subset K'$, such that $(g,e^{iz})=f(z)$ for all $z \in T(L')$ a *spectral function* for f.

Before we consider the existence and uniqueness of such a spectral function, we give a number of ancillary definitions.

Let $J(-iD)$ be the non-local differential operator then

$$J(-iD_\zeta)e^{iz\zeta}=\sum_{\alpha\geq 0} a_\alpha(-i)^{|\alpha|}(iz)^\alpha e^{iz\zeta}=J(z)e^{iz\zeta}, \text{ i.e.,}$$

$J(-iD)e^{iz}=J(z)e^{iz}$. Further, suppose $J(z)=e^{-az}$ where $a \in R^n$ then $e^{iaD}\phi(\zeta) =\sum_{\alpha\geq 0}(ia)^\alpha\phi^{(\alpha)}(\zeta)/\alpha! =\phi(\zeta+ia)$, wher ϕ is a function which is holomorphic in a sufficiently large tube region.

Definition 4.2. Let K be a convex compact subset of R^n with nonempty interior. We denote by $\Lambda(K)$ the Banach space of all continuous functions $\phi(\xi)$ for which

$$\|\phi\|_{K_w}=sup\{e^{W_K(\xi)}|\phi(\xi)| : \xi \in R^n\}<\infty,$$

with the topology defined by the norm $\|\cdot\|_{K_w}$.

Definition 4.3. We shall say that the pair K,K' is *$\mathscr{L}$-admissible* if for any $L \supset K$, $L' \Subset K'$ there exists an entire function $J(z)$ such that

A. $J(-iD) : Q_w(T(L) ; L') \to \Lambda(L')$ is continuous, linear,

B. There exist $K_j \supset K$ and $K'_j \supset K'$ such that

$1/J(z) \subset Q_s(T(K'_j) ; K_j)$.

Proposition 4.4. Suppose $J(z)=\sum_{\nu=1}^{n} exp(-a_\nu z)$, $a_\nu \in K (\nu=1,2,\ldots,n)$, then $J(-iD)$ is non-local and

$$J(-iD) : Q_w(T(K) ; K') \to \Lambda(K')$$

is a continuous linear differential operator.

Proof. Suppose $\phi \varepsilon Q_w(T(K) : K')$, then $J(-iD)\phi(\xi)=\sum_{\nu=1}^{n} e^{ia_\nu D}\phi(\xi)= \sum_{\nu=1}^{n} \phi(\xi+ia_\nu)$ and therefore

$$\| J(-iD)\phi \|_{K_w} = sup\{e^{W_{K'}(\xi)} | \sum_{\nu=1}^{n} \phi(\xi+ia_\nu) | : \xi \in R^n\}$$

$$\leq n \cdot sup \{e^{W_{K'}(\xi)} | \phi(\zeta) | : \zeta \in T(K)\} = n \| \phi \|_{K_w}$$

We now give a sufficient condition for a pair K, K' to be $\mathscr{L}$ -admissible.

Proposition 4.5. Let K be a convex compact subset of R^n with nonempty interior. Suppose there exists a positive number δ and K' convex compact subset of R^n with nonempty interior such that

$$| K \cdot K' | \leq \frac{1}{2} - \delta$$

then K, K' is an $\mathscr{L}$-admissible, where $K\cdot K'=\{x\xi : x \in K, \xi \in K'\}$ and $|K\cdot K'|\leq\rho$ if and only if $|x\cdot\xi|\leq\rho$ for all $x \in K, \xi \in K'$.

Proof. Suppose we are given a pair $L\supset K$, $L'\Subset K'$, Since K is convex compact, there exist points $a_1,\ldots,a_N \in L\setminus K$ such that the polyhedron

$K_j=\mathrm{ch}\{a_1,\ldots,a_N\}$ satisfies these conditions ;

1) $L\supset K_j\supset K$;

2) There exist positive number δ' and $K_j'\supset K'$ such that

$$|K_jK_j'|\leq\frac{\pi}{2}-\delta'.$$

We set $J(z)=\sum_{\nu=1}^{N} e^{-a_\nu z}$. Then, by Proposition 4.4,

$J(-iD) : Q_w(T(L) ; L')\to\Lambda(L')$ is continuous, linear, and hence condition A is satisfied. Suppose $z \in T(K_j')$ then

$$|J(z)| = |\sum_{\nu=1}^{N} e^{a_\nu z}|\geq Re\sum_{\nu=1}^{N} e^{-a_\nu z}=\sum_{\nu=1}^{N} e^{-a_\nu x}\cos a_\nu y$$

$$\geq\sum_{\nu=1}^{N} e^{-a_\nu x}\sin\delta'>0,$$

where $z=x+iy$. Therefore

$$\|\frac{1}{J}\|K_j',\ K_j=\sup\ \{e^{-W_{K_j}(x)}|\frac{1}{J(z)}| : z=z=x+iy\in T(K_j')\}$$

$$\leq\sup_{x\in R^n}\frac{e^{-W_{K_j}(x)}}{\sin\delta'\sum_{\nu=1}^{N}e^{a_\nu x}}\leq\frac{1}{\sin\delta'}\quad ,$$

since $e^{-W_{K_J}(x)} \leq \sum_{\nu=1}^{N} e^{-a_\nu x}$ for all $x \in R^n$.

5. A spectral function in the case of $\mathscr{L}$-admissible pairs.

Proposition 5.1. Let K,K' be an $\mathscr{L}$-admissible pair, $f \in Q_w(T(K') ; K)$, J is an entifre function with the properties A and B of the definition of $\mathscr{L}$-admissibility and define

$$f_J(z)=\frac{f(z)}{J(z)}$$

then $f_J \in Q_s(T(K') ; \bar{U}_\varepsilon)$ for sufficiently small $\varepsilon>0$.

Proof. By property B, there exist $K_J \supset K$, $K'_J \supset K'$ such that $\frac{1}{J(z)} \in Q_s(T(K'_J) ; K_J)$, and since $f(z) \in Q_w(T(K') ; K)$,

$$\left|\frac{1}{J(z)}\right| \leq C\ e^{W_{K_J}(x)} \text{ for some constant } C>0,\ z \in T(K'_J), \text{ and}$$

$$|f(z)| \leq M e^{-W_K(x)} \text{ for some constant } M>0,\ z \in T(K').$$

$$\|f_J\|_{K',\bar{U}_\varepsilon} = \sup\ \{e^{-W_{\bar{U}_\varepsilon}(x)}\ \left|\frac{f(z)}{J(z)}\right|\ : z=x+iy \in T(K')\}$$

$$\leq CM\ \sup\{e^{-W_{\bar{U}_\varepsilon}(x)-W_K(x)+W_{K_J}(x)} : x \in R^n\}.$$

By property e) of the nega-support function[11], for every $K \Subset K_J$ there exists $\delta(K)>0$ such that $W_{K_J}(x)-W_K(x) \leq -\delta\,|x|$. Hence

$$-W_{\bar{U}_\varepsilon}(x)-W_K(x)+W_{K_J}(x) \leq -W_{\bar{U}_\varepsilon}(x)-\delta\,|x|$$

$$\leq \varepsilon |x| - \delta |x| = (\varepsilon - \delta) |x| .$$

Therefore, if $\varepsilon < \delta$ then $\| f_j \|_{K', \bar{U}_\varepsilon}$ is finite.

Proposition 5.2. Let K, K' be an $\mathscr{L}$ admissible pair, $f \in Q_w(T(K') ; K)$, $L \supset K$, $L' \Subset K'$ then there exists a spectral function $g \in Q'_w(T(L) ; L')$ such that

a) $f(z) = (g, e^{iz})$ for every z in $T(L')$,

b) $\| g \| \leq C \| f \|_{K', K_w}$,

where the constant C does not depend on f.

We set $g_j(\xi) = \mathscr{F}^{-1}[f_j](\xi)$ where f_j is the function defined in Proposition 5.1, and then $g_j(\xi) \in Q_s(T(\bar{U}_{\varepsilon'}) ; M')$, where $0 < \varepsilon' < \varepsilon$, $L' \Subset M' \Subset K'$, and $f_j(z) = \mathscr{F}[g_j](z)$ and $\| g_j \|_{\bar{U}_{\varepsilon'}, M'} \leq C \| f_j \|_{K', \bar{U}_\varepsilon}$ for some constant $C > 0$.

In particular, the function g_j defines a regular functional on $\Lambda(L')$ that acts in accordance with the formula

$$(g_j, \phi) = \int g_j(\xi) \phi(\xi) d^n \xi, \text{ and}$$

$$(g_j, e^{iz}) = \mathscr{F}[g_j](z) = f_j(z), \text{ for every } z \in T(L'),$$

and $\| g_j \|_{\Lambda'(L')} \leq C' \| g_j \|_{\bar{U}_{\varepsilon'}, M'} \leq C'' \| f_j \|_{K', \bar{U}_\varepsilon}$ where C' and C'' are some constants.

We define the functional $g \in Q'_w(T(L)\,;L')$ by the equation $(g,\phi)=(g_J, J(-iD)\phi)$, for every $\phi \in Q_w(T(L)\,;L')$.
By property A of the function J, this definition is correct.

We verify that g has the required properties a) and b). Suppose $z \in T(L')$, then

$$(g,e^{iz\xi})=\int g_J(\xi)[J(-iD)e^{iz\xi}]d^n\xi=\int g_J(\xi)J(z)e^{iz\xi}d^n\xi$$

$$=J(z)(g_J,e^{iz\xi})=J(z)f_J(z)=f(z),$$

and a) is proved.
Further, by virtue of the estimates obtained above, for any $\phi \in Q_w(T(L)\,;L')$,

$$|(g,\phi)| \leq \| g_J \|_{\Lambda'(L')} \| J(-iD)\phi \|_{\Lambda(L')}$$
$$\leq C \| f_J \|_{K',\bar{U}_\varepsilon} \| J(-iD) \| \ \| \phi \|_{L,L'_w}, \text{ for some constant}$$

$C>0$, where $\| J(-iD) \|$ is the norm of the operator $J(-iD)$.
However,

$$\| f_J \|_{K',\bar{U}_\varepsilon}=sup\{e^{-W_{\bar{U}_\varepsilon}(x)} \left|\frac{f(z)}{J(z)}\right| : z=x+iy \in T(K')\}$$

$$\leq \sup_{x\in R^n} \{e^{\varepsilon|x|} \| f \|_{K',K_w} e^{-W_K(x)} \| \frac{1}{J} \|_{K'_J,K_J} e^{W_{K_J}(x)}\}$$

$$= \| f \|_{K',K_w} \| \frac{1}{J} \|_{K'_J,K_J} \sup_{x\in R^n} \{e^{\varepsilon|x|-W_K(x)+W_{K_J}(x)}\}$$

$$= \|f\|_{K',K_w} \|\frac{1}{J}\|_{K'_J,K_J}$$

Since $\varepsilon|x|-W_K(x)+W_{K_J}(x)\leq 0$ for sufficiently small ε.
Therefore

$|(g,\phi)|\leq C\|f\|_{K',K_w}\|\phi\|_{L,L'_w}$, where the constant C does not depend on f or ϕ, so that property b) also holds.

To study the uniqueness of the spectral function, we require ancillary assertions about integration under the symbol of an analytic functional.

Proposition 5.3. Suppose $g\in Q'_w(T(K);K')$. F is a compact set in R^n, $y\in L'\Subset K'$, $\psi\in\mathscr{H}(T(K'))$ then

$$\int_F (g,e^{iz})\psi(z)d^nx=(g(\zeta),\int_F e^{iz\zeta}\psi(z)d^nx),\quad (z=x+iy).$$

Proof. Suppose $\sum_{\nu=1}^{N}[z^{iz_\nu}\psi(z_\nu)]$ mes F_ν $(z_\nu=x_\nu+iy)$ is an integral sum for the integral $\int_F e^{iz}\psi(z)d^nx$, where $\{F_\nu:\nu=1,2,\ldots,N\}$ is a partitioning of F with mesh δ(i.e., $\delta=sup\{d(F_\nu):\nu=1,2,\ldots,N\}$, where $d(F_\nu)$ is diameter of F_ν), mes F_ν is the Lebesgue measure of F_ν. Consider the difference

$$\triangle(\zeta)=\sum_{\nu=1}^{N}[e^{iz_\nu}\ \psi(z_\nu)]\ \text{mes}\ F_\nu-\int_F e^{iz\zeta}\psi(z)d^nx$$

$$\sum_{\nu=1}^{N}\int_{F_\nu}[e^{iz_\nu\zeta}\ \psi(z_\nu)-e^{iz\zeta}\psi(z)]d^nx.$$

By means of the first mean value theorem, we obtained

$$|\triangle(\zeta)| \leq \sum_{\nu=1}^{N} |e^{iz_\nu\zeta}\psi(z_\nu)-e^{iz'_\nu\zeta}\psi(z'_\nu)| \text{ mes } F_\nu (z'_\nu=x'_\nu+iy)$$

$$\leq \text{mes} F \sup\{|e^{iz'\zeta}\psi(z')-e^{iz''\zeta}\psi(z'')| : z'=x'+iy,\ z''=x''+iy$$

$$|x'-x''| \leq \delta,\ x',\ x'' \in F\}$$

Fixing a sufficiently small $r>0$ and using the integral Cauchy formula, we obtain

$|\triangle(\zeta)| \leq \text{mes } F \frac{2}{r} \sup\{|e^{iz\zeta}\psi(z)| : z \in F_r+iL'_r\}\delta$, for every $\delta \leq \frac{r}{2}$ (we recall that F_r and L'_r are real r-neighborhood of the sets F and L'). Therefore

$$\|\triangle\|_{K,K'_w} = \sup\{e^{W_{K'}(\xi)}|\triangle(\zeta)| : \zeta \in T(K)\}$$

$$\leq \sup_{\zeta\in T(K)}\{e^{W_{K'}(\xi)} \text{ mes } F \frac{2}{r} \sup_{z\in F_r+iL'_r}|e^{iz\zeta}| \sup_{z\in F_r+iL'_r}|\psi(z)|\}\delta$$

$$\leq [\text{mas } F\frac{2}{r} \sup_{z\in F_r+iL'_r}|\psi(z)| \sup_{x\in F_r} e^{-W_K(x)} \sup_{\xi\in R^n} e^{W_{K'}(\xi)-W_{L'_r}(\xi)}]\delta$$

$$= C\delta, \text{ for every } \delta \leq \frac{r}{2},$$

since $W_{K'}(\xi)-W_{L'_r}(\xi) \leq 0$ for sufficiently small r.
Thus $\|\triangle\|_{K,K'_w} \to 0$ in the limit $\delta \to 0$. This means that the integral sums converge to the integral in topology of $Q_w(T(K) ; K')$ and therefore, one can integrate under the functional sign, from which the required equation then follows.

Theorem 5.4. Suppose $g \in Q'_w(T(K);K')$, $\psi \in Q_s(T(K');L)$, $L \supset K$, $z=x+iy \in T(K')$ then

$$\int (g, e^{iz})\psi(z)d^n x = (g, \mathscr{F}[\psi]).$$

For suppose $R>0$, we set

$$\triangle_R(\zeta) = \mathscr{F}[\psi](\zeta) - \int_{|x|\leq R} e^{i\zeta z}\psi(z)d^n x = \int_{|x|>R} e^{i\zeta z}\psi(z)d^n x.$$

Then we can show that $\triangle_R(\zeta) \in Q_w(T(K);K')$ and $\|\triangle_R\|_{K,K'_w} \to 0$ as $R\to\infty$.

This means that $\int_{|x|\leq R} e^{i\zeta z}\psi(z)d^n x \to \mathscr{F}[\psi](\zeta)$ in the topology of $Q_w(T(K);K')$ as $R\to\infty$, so that

$$\lim_{R\to\infty} (g(\zeta), \int_{|x|\leq R} e^{i\zeta z}\psi(z)d^n x) = (g, \mathscr{F}[\psi]).$$

By the Proposition 5.3, $\int_{|x|\leq R} (g, e^{iz})\psi(z)d^n x = (g(\zeta), \int_{|x|\leq R} e^{i\zeta z}\psi(z)d^n x)$,

we obtain the required equation.

Theorem 5.5. Suppose $g \varepsilon Q'_w(T(K);K')$ and $(g, e^{iz})=0$ for all $z \in T(\{y\}) = R^n + iy$ for some $y \varepsilon K'$. Then $g=0$ on $\overleftarrow{Q}_s(C^n)$ and hence $g=0$ on any space $Q_w(T(L);L')$ with $L \supset K$, $L' \subset K'$.

Proof. For any function $\psi \in \overleftarrow{Q}_s(C^n)$, $z=x+iy$, $(g, \mathscr{F}[\psi]) = \int (g, e^{iz})d^n x = 0$ by Theorem 5.4. Since $\mathscr{F}(\overleftarrow{Q}_s(C^n)) = \overleftarrow{Q}_s(C^n)$, $g=0$ on $\overleftarrow{Q}_s(C^n)$. Because $\overleftarrow{Q}_s(C^n)$ is dense in $Q_w(T(L);L')$ in the topology of $Q_w(T(K):K')$ if $L \supset K$, $L' \Subset K'$, $g=0$ on $Q_w(T(L);L')$.

By the above Theorem 5.5 and the Hahn-Banach theorem the linear hull of the set $\{e^{i(x+iy)}:x\in R^n\}$ for fixed $y\in L'$ is dense in $Q_w(T(L);L')$ in the topology of $Q_w(T(K);K')$ for all $L \supset K$, $L' \Subset K'$.

Theorem 5.6. Let g', $g'' \in Q'_w(T(L);L')$ be two spectral functions for $f\varepsilon Q_w(T(K');K)$, where $L \supset K$, $L' \Subset K'$ then $g'=g''$ on any space $Q_w(T(M);M')$ with $M \supset L$, $M' \Subset L'$.

To prove this, it sufficient to set $g=g'$-g' and use Theorem 5.5.

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