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A Remark on the Existence of Periodic Solutions to the First Order Ordinary Differential Equations

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I. Introduction

It is know that the system of the form

(E) $$\dot{x}=f(t, x)$$

where $f: RxR^n \rightarrow R^n$ is continuous and T-periodic with respect to t for some positive constant T, has at least one T-periodic solution if we assume

(H_1) The solution to IVP for (E) are unique

(or $\langle f(t, x_1)-f(t, x_2), x_1-x_2 \rangle < 0$, $t \in [0, T]$, x_1, $x_2 \in R$ with $\|x\|=r$ for some $r>0$),

(H_2) $\langle f(t, x), x \rangle < 0$ for all $(t, x) \in RxR^n$ with $\|x\|=r$ for some $r>0$.

It is natural to ask whether we can find an appropriate sign condition which is independenpent of (H_1), (H_2) and still guarantees the existence of T-periodic solution to the system (E). The answer is affirmative when we replace (H_2) by a generalized sign condition far away from the origin without assuming (H_1).

In section 1, we give the answer to our question in BVP view. More precisely, we investigate the existence of T-periodic solutions to the BVP

$$\dot{x} = f(t, x)$$
$$x(0) = x(T)$$

,where $x = x(t)$, $f : RxR^n \to R^n$ is T-periodic with respect to t and continuous(or f is a Caratheodory function having sublinear growth in x). The proof is based on Leray-Schauder's continuation theorem.

In section 2, we extend our result to the delay functional differential equations. More precisely, we devote ourselves to prove the existence of T-periodic solutions to the BVP

$$\dot{x} = f(t, x_t)$$
$$x_0 = x_T$$

, where $x = x(t)$, $x_t : [-r, 0] \to R^n$, $x_t(s) = x(t+s)$ and $f : RxC_r \to R^n$ is a continuous function and takes bounded sets into bounded sets. Here r is a non-negative constant and C_r is the Banach space of continuous mappings $h : [-r, 0] \to R^n$ with the norm

$$\| h \| = \sup_{s \in [-r,0]} | h(s) | .$$

The proof is based on Mawhin's continuation theorem.

Our results are related to these results in [2], [3] which are derived from the method of guiding functions.

II. First Order Ordinary Differential Equations.

Let C_T be the Banach space of mappings $x : R \to R^n$ which are continuous and T-periodic with the norm

$$\|x\|_{C_T} = \sup_{t\in R} \|x(t)\|$$

$\|x(t)\|$ is the Euclidean norm of $x(t)$. Let ϕ_i and μ_i, i=1, 2,⋯,s be linear independent solutions to the T-periodic, homogeneous differential equations $\dot{x}=A(t)x$ and its adjoint $\dot{y}=-A^*(t)y$ with $A: R\to R^n$ continuous and T-periodic, respectively. By Gram Schmidt procedure, we may assume

$$\langle\phi_i,\phi_j\rangle = \langle\mu_i,\mu_j\rangle = \delta_{ij},\quad 1\leq i,j\leq s.$$

Define

$$P: C_T\to C_T \text{ by } Px= \sum_{1\leq i\leq s}\langle x,\phi_i\rangle\phi_i$$
$$Q: C_T\to C_T \text{ by } Px= \sum_{1\leq i\leq s}\langle x,\mu_i\rangle\mu_i .$$

Then they are projections.

Proposition. Suppose $A(t)$ and $b(t)$ are continuous and T-periodic on R. The euqation

(II.1) $\dot{x}=A(t)x+B(t)$

has a T-periodic solution if and only if

(II.2) $Qb=0.$

If (II.2) is satisfied, then (II.1) has unique T-periodic solution such that $Px=0$.

Now let $C_{TI-P}=\{x\in C_T \mid Px=0\}$
$C_{TI-Q}=\{x\in C_T \mid Qx=0\}$.

Define $K: C_{TI-Q}\to C_{TI-P}, b\to x$, where x is a solution to (II.1). Then K is well-defined, linear and $K(0)=0$. And since $I-Q: C_T\to C_{TI-Q}$

is linear, $K(I-Q):C_T\rightarrow C_{TI-P}$ is well-defined, linear and bounded. Moreover, $K(I-Q):C_T\rightarrow C_T$ is a compact operator.
You may find the above mentioned results in [3].

Lemma II.1 Let $F:[0,T]\mathrm{x}R^n\rightarrow R,^n(t,x)\rightarrow F(t,x)$ be contiouous function, then $H:C_T\rightarrow C_T,\ x\rightarrow Hx=F(\cdot,x(\cdot))$ is a continuous and maps bounded sets into bounded sets.

Lemma II.2 If A is a positive definite operator, then there is $c>0$ such that $\langle Ax,x\rangle>c\|x\|^2$ for all $x\in R^n$,

Theorem II.1 Let $f:R\mathrm{x}R^n\rightarrow R^n$ be continuous and T-periodic function with respect to t. Let $A:R^n\rightarrow R_n$ be symmetric and positive definite linear operator and $\langle f(t,x),\ Ax\rangle>0$ for $\|x\|>r$ for some $r>0$. Then BVP

(E) $\dot{x}=f(t,\ x)$

(B) $x(0)=x(T)$

has at least one solution.

Proof Let $D(L)=C_T\cap C^1[0,T]$. Define an operator $L:D(L)\subset C_T\rightarrow C_T$ by $Lx=\dot{x}-x$ for $x\in D(L)$, then $\dot{x}-x=0$ has only trivial T-periodic solution which implies $P=Q=0$. Hence for each $f\in C_T$ the T-periodic solution to $\dot{x}=x+f$ exists uniquely.
Therefore $L^{-1}:C_T\rightarrow C_T,\ f\rightarrow x$ exists and is a compact operator. Now consider a substitution operator

$$N:C_T\rightarrow C_T,\ x\rightarrow -x(\cdot)+f(\cdot,x(\cdot)).$$

Then N is continuous and maps bounded sets into bounded sets. Therefore, $x\in C_T$ is a solution to the BVP (B) (E) if and only if

$x \in D(L)$ and x satisfies

(II.3) $\quad Lx = Nx$, or equivalently

(II.4) $\quad x = L^{-1}Nx$

Since L^{-1} is a completely continuous and N is continuous and maps bounded sets into bounded sets, the composition $L^{-1}N : C_T \to C_T$ is contiouous and compact

By using Leray-Schauder's degree arguement, if all possible solution x to the family of equations

(II.5) $\quad x = L^{-1}Nx, \ 0 \leq \lambda \leq 1,$

are buonded in C_T independently of λ, then (II.4) has a solution. If (x,λ) solves (II.5), then (x,λ) solves

(II.6) $\quad Lx = \lambda Nx, \ 0 \leq \lambda \leq 1,$

and x is a solution to the T-periodic BVP of the equation

(II.7) $\quad \dot{x} = (1-\lambda)x + \lambda f(t,x), \ 0 \leq \lambda \leq 1$

When $\lambda = 0$ by our assumption, we have only trivial T-periodic solution. Thus the proof will be completed if we show that the solution to (II.6), for $0 < \lambda \leq 1$, are bounded in C_T independently of λ. To this end, define $\phi : R^n \to R$ by $\phi(x) = \langle Ax, x \rangle$. Let $M = \sup_{\|x\| \leq r} \phi(x)$, then since $\lim_{\|x\| \to +\infty} \phi(x) = \infty$, for $M_0 > M$, there $r_0 > r$ such that $\phi(x) > M_0 > 0$ whenever $\|x\| > r_0$.

We prove that for any possible T-periodic solution x to (II.7), we have

(II.8) $\|x(t)\| \leq r_0$ for all $t \in [0,T]$

To do this, define $v: R \to R$, $t \to \phi(x(t))$, then v is of class C^1 and T-periodic and such that

(II.9) $$\dot{v}(t) = 2\langle Ax(t), \dot{x}(t)\rangle = 2(1-\lambda)\langle Ax(t), x(t)\rangle + 2\lambda\langle Ax(t), f(t, x(t))\rangle,$$

for all $t \in \mathrm{R}$. For every value t_0 of t such that

$$v(t_0) = \sup_{t \in R} v(t) = \sup_{t \in [0,T]} v(t),$$

we have $\dot{v}(t_0) = 0$, since v can be estended on the whole of R. If $\|x(t_0)\| > r$, then $\langle f(t_0), Ax(t_0)\rangle > 0$. Thus

$$\dot{v}(t_0) = 2(1-\lambda)\langle Ax(t_0), x(t_0)\rangle + 2\lambda\langle Ax(t_0), f(t_0, x(t_0))\rangle > 0$$

which is impossible. Hence $\|x(t_0)\| \leq r$.

If there exists $t_1 \in [0,T]$ such that $\|x(t_1)\| > r_0$, then $v(t_1) = \langle Ax(t_1), x(t_1)\rangle > M_0$ and so $M_0 < v(t_1) \leq v(t_0) = \langle Ax(t_0), x(t_0)\rangle \leq \sup_{\|x\| \leq r} \langle Ax(t), x(t)\rangle = M$ which is a contradiction. Hence we have $\|x(t)\| \leq r_0$ for all $t \in [0,T]$ for every possible T-periodic solution to (II.7).

So we have that every solution (x, λ) to (II.5) has an a' priori hound in C_T independently of λ. Therefore, by Leray-schauder's continuation theorem, $x = L^{-1}Nx$ has a solution, or $\dot{x} = f(t, x)$ has a solution in C_T.

Corollary II.1 Let $f: RxR^n \to R^n$ be continuous and T-periodic function with respect to t. Let $\langle f(t, x), x\rangle > 0$ for $\|x\| \geq r$ for some $r > 0$. Then BVP (E) (B) has at least one solution.

Theorem II.2. Let $f: RxR^n \to R^n$ be continuous and T-periodic function with respect to t with $\|f(t, x)\| \leq \alpha\|x\| + \beta$ for some α,

$0<\alpha<1/T$, $\beta>0$ for all $(t, x)\in RxR^n$. Let $A: R^n\to R^n$ be symmetric linear operator and has no eigenvalue with zero real part, and $\langle f(t, x), Ax\rangle>0$ for $\|x\|>r$ for some r>0. Then BVP (B) (E) has at least one solution in C_T.

Proof Let $D(L)=C_T\cap C^1[0, T]$. Define an operator $L: D(L)\subseteq C_T \to C_T$ by $Lx=\dot{x}-\varepsilon Ax$, where ε such that $\varepsilon T\|A\|+\alpha T<1$, for $x\in D(L)$, then for each $f\in C_T$, the T-periodic solution x to $\dot{x}=\varepsilon Ax+f$ exists uniquely. Therefore $L^{-1}: C_T\to C_T, f\to x$ exists and is a compact operator. Now we consider a substitution operator

$$N: C_T\to C_T,\ x\to -\varepsilon Ax(\cdot)+f(\cdot,\ x(\cdot))$$

Then N is continuous and maps bounded sets into bounded sets. Therefore, $x\in C_T$ is a solution to the BVP (B) (E) if and only if $x\in D(L)$ and x satisfies

(II.10) $Lx=Nx$, or

(II.11) $x=L^{-1}Nx$.

Since L^{-1} is a completely continuous and N is continuous and maps bounded sets into bounded sets, the composition $L^{-1}N: C_T\to C_T$ is continuous and compact. By using Leray-Shcauder's degree arguement, if all solution x to the family of equations.

(II.12) $x=\lambda L^{-1}Nx,\ 0\leq\lambda\leq 1,$

are bounded in C_T independently of λ, then (II.10) has a solution. If (x, λ) solves (II.12), then (x, λ) solves

(II.13) $Lx=\lambda Nx,\ 0\leq\lambda\leq 1,$

and x is solution to the T-periodic BVP of the equation.

(II.14) $\dot{x}=(1-\lambda)\varepsilon Ax+\lambda f(t, x), \ 0 \leq \lambda \leq 1.$

If $\lambda = 0$ we have only trivial T-periodic solution. Thus, the proof will be completed if we show that the solution to (II.12), for $0<\lambda \leq 1$, are bounded in C_T independently of λ. To this end, let (x, λ) be any solution to (II.13) with $0 < \lambda \leq 1$ then

$$\begin{aligned}\|\dot{x}\| &=(1-\lambda)\|\varepsilon Ax\| +\lambda\|f(t, x)\| \quad (0 < \lambda \leq 1)\\ &\leq \|\varepsilon Ax\| + \|f(t, x)\|\\ &\leq \varepsilon\|A\| \ \|x\| +\alpha\|x\| +\beta\\ &=(\varepsilon\|A\| +\alpha)\|x\| +\beta.\end{aligned}$$

If $\|x(t)\| \geq r$ for all $t \in [0, T]$, then

$$\begin{aligned}0&= \int_0^T \langle \dot{x}(t), Ax(t)\rangle dt\\ &=(1-\lambda)\varepsilon\int_0^T \langle Ax(t), Ax(t)\rangle dt+\lambda \int_0^T \langle f(t, x(t)), Ax(t)\rangle dt>0\end{aligned}$$

which is impossible. Hence there is a $t_0 \in [0, T]$ such that $\|x(t_0)\| < r$. Since $x(t)=x(t_0)+ \int_{t_0}^t \dot{x}(t)dt$, $\|x\| \leq r+ \int_0^T \|\dot{x}\| dt=r+T\|\dot{x}\|$. Therefore,

$$\begin{aligned}&\|x\| \leq r+[\varepsilon T\|A\| +\alpha T]\|x\| +\beta T, \text{ or}\\ &[1-\varepsilon T\|A\| -\alpha T]\|x\| \leq r+\beta T.\end{aligned}$$

Since $\varepsilon T\|x\| +\alpha T < 1$, we have

$$\|x\| \leq (r+\alpha T)/(1-\varepsilon T\|A\| -\alpha T).$$

Hence, we have that every solution (x, λ) to (II.12) has an a priori bound in C_T independently of λ. Therefore, by the Leray-Schauder's continuation Theorem, $x=L^{-1}Nx$ has a solution, or $\dot{x}=f(t, x)$ has a solution in C_T.

Corollary II.12 Let $f: RxR^n \to R^n$ be continuous function and

T-periodic function with respect to t with $\| f(t, x) \| \leq \alpha \| x \| + \beta$ for some α, $0 < \alpha < 1$, $\beta > 0$ for all $(t, x) \in R x R^n$. And $\langle f(t, x), x \rangle 0$ for $\| x \| > r$, for some $r > 0$.

Then BVP (B) (E) has at least one solution in C_T.

Example $$\dot{x} = ax + bx^3 + e(t)$$ $$x(0) = x(T),$$

where $e : R \to R$ is continuous, T-periodic and $b > 0$, has at least one T-periodic solution.

III. First Order Ordinary Delay Functional Differential Equations

Let us denote by C_T the Banach space of continus and T-periodic mappings $x : R \to R^n$ with the norm $\| x \|_{C_T} = \sup_{t \in R} \| x(t) \|$ where $\| \cdot \|$ is the Euclidean norm in R^n. For some $r > 0$ let C_r be the Banach space of continuous mapping $\phi : [-r, 0] \to R^n$ with the norm $\| \phi \|_{C_r} = \sup_{\theta \in [-r, 0]} \| \phi(\theta) \|$. When $r = 0$, C_r is naturally identified to R^n If $x \in C_T$ and $t \in T$, we shall denote by x_t the element of C_r defined by

$$x_t : [-r, 0] \to R^n, \quad \theta \to x(t + \theta).$$

We note that,

$$\| x_t \|_{C_r} = \sup_{\theta \in [-r, 0]} \| x(t + \theta \| \leq \sup_{t \in R} \| x(t) \| = \| x \|_{C_T}.$$

When $r = 0$ the mapping x_t will be naturally identified with the element $x(t)$ of R^n. Moreover we shall sometimes identify, without further comment, a constant mapping in C_T or C_r with the element of R^n

given by its constant value.

Let $f: RxC_r \to R^n,\ (t,\ \phi) \to f(t,\ \phi)$

be T-periodic with respect to t, continuous and take bounded sets into bounded sets. Let us consider the functional differential equation.

(III.1) $\dot{x}=f(t,\ x_t).$

If we define the Banach space by $X=\{x \in C_r : x_0=x_T\}$ and

$$DomL=X \cap C^1[0,\ T] \cap C_T$$
$$L: DomL \to C_T,\ x \to \dot{x},$$
$$N: C_T \to C_T,\ x \to f(\cdot,\ x),$$

then $KerL=R^n\ ImL=\{y \in C_T : \int_0^T y(s)ds=0\}.$

Let us introduce the continuous projectors

$$P: C_T \to C_T,\ x \to x(0)$$
$$Q: C_T \to C_T,\ x \to 1/T \int_0^T x(s)ds.$$

Then for each $x \in C_T$

$$\| Qx \|_{C_T} < \| x \|_{C_T}$$

and ImQ is the subspace of C_T of constant mappings, and the following sequence is exact;

$$C_T \xrightarrow{P} DomL \subset C_T \xrightarrow{L} C_T \xrightarrow{Q} C_T$$

which implies

$$KerL=ImP,\ ImL=KerQ,$$

and

$$C_T = ImP \oplus KerP = KerL \oplus KerP, \quad C_T = ImQ \oplus KerQ = ImQ \oplus ImL$$

as topological sums.
Thus we have $C_T / ImL \simeq ImQ$,

$ImP = \{x(0) : x \in P\} = R^n$,

$ImQ = \{1/T \int_0^T x(s)ds \,;\, x \in C_T\} = R^n$.

$dim\ KerL = n = dim\ ImQ = dim\ C_T/ImL = dim\ CoKerL < \infty$, L is linear and ImL is closed in C_T. Hence L is Fredholm mapping of index zero and there exists an isomorphism

$$J : ImQ \rightarrow KerL.$$

If we consider the restriction

$$L_P = L \mid_{Dom \cap KerP} : DomL \cap KerP \rightarrow ImL,$$

then L_P is bijective, so that its algebraic inverse

$$K_P = L_P^{-1} : ImL \rightarrow DomL \cap KerP$$

is defined and $K_P(y)(t) = x(t) = \int_0^T y(s)ds$

We will denote $K_{PQ} : C_T \rightarrow DomL \cap KerP$ the generalized inverse of L defined by $K_{PQ} = K_P(I-Q)$.
Then K_{PQ} is a compact operator by Arzela-Ascoli theorem. $K_{PQ}N$ takes bounded sets into relatively compact sets since N takes bounded sets into bounded sets. You may find the following Lemma in Mawhin [1], Mawhin and Gains [2].

Lemma III.1 With the assumption and notations above, N is L-compact on each bounded subset of C_T.

Theorem III.1 Let $f : RxC_r \to R^n$ be T-periodic with respect to t, continuous and takes bounded sets into bounded sets. Let $A : R^n \to R^n$ be a symmetric and positive definite linear operator such that $\langle f(t, x_t), Ax \rangle > 0$, $\| x \| \geq r$ for some $r > 0$. Then BVP

(E) $\dot{x} = f(t, x_t)$

(B) $x_0 = x_T$

has at least on solution

Proof. We will apply Mawhin's continuation theorem to our proof. Now it is easy to see $x \in C_T$ is a solution BVP (E) (B) if and only if $x \in DomL$ and

(III.1) $$Lx = Nx.$$

Since L is a Fredholm mapping of index zero and N is L-compact, by Mawhin's continuation theorem if there exits a bounded open set G in C_T such that

(a) for each $\lambda \in]0, 1[$, every solution x of
$Lx = \lambda Nx$
is such that $x \in \partial G$.

(b) $QNx \neq 0$ for each $x \in KerL \cap \partial G$ and
$d(JQN|_{KerL}, G \cap KerL, 0) \neq 0$
,where d is the Brouwer topological degree.

Then the equation $Lx = Nx$ has at least one solution in $DomL \cap G$.

Now we prove (a). For this purpose, let (x, λ) be any solution to

(III.2) $$Lx = \lambda Nx,$$

then (x, λ) is a solution to BVP

(E) $$\dot{x}=\lambda f(t, x_t)$$

(B) $$x_0=x_T$$

Let $M=\sup_{\|x\|\leq r}\langle Ax, x\rangle$, then since $\lim_{\|x\|\to\infty}\langle Ax, x\rangle=\infty$,
for $M_0>M$, there exists $r_0>r$ such that $\langle Ax, x\rangle>M_0$ whenever $\|x\|>r_0$.
Let us define $v: R\to R$ by

$$v(t)=\langle Ax(t), x(t)\rangle \text{ for all } t\in R$$

Then, v is of class C^1 and T-periodic such that

$$\begin{aligned}\dot{v}(t)&=2\langle Ax(t), \dot{x}(t)\rangle\\&=2\lambda\langle Ax(t), f(t, x_t)\rangle \text{ for all } t\in R\end{aligned}$$

For every value t_0 of t such that

$$v(t_0)=\sup_{t\in R}v(t)=\sup_{t\in[0,T]}v(t),$$

we have $\dot{v}(t_0)=0$ If $\|x(t_0)\|>r$, then $\langle f(t_0, x_{t_0}), Ax(t_0)\rangle>0$.

Thus

$$\dot{v}(t_0)=2\lambda\langle Ax(t_0), f(t_0, x_{t_0})\rangle>0,$$

which is impossible. Hence $\|x(t_0)\|<r$.

If there exists t_1 in $[0,T]$ such that $\|x(t_1)\|>r_0$, then

$$v(t_1)=\langle Ax(t_1), x(t_1)\rangle>M_0$$

and so

$$M_0<v(t_1)\leq v(t_0)=\langle Ax(t_0), x(t_0)\rangle\leq\sup_{\|x\|<r}\langle Ax,x\rangle=M$$

which is impossible. Hence, we have $\|x(t)\|\leq r_0$ for all $t\in[0,T]$, i.e.,

$$\|x\| = \sup_{t\in[0,T]} \|x(t)\| < r_0$$

for every possible solution to (III.2). Therefore every solution (x, λ) of (III.2) is such that $x \in \partial G$ where G is an open ball in C_T with radious $p > r_0$ and centered at origin.

Now we will show that the condition (b) is satisfied, Since $\langle f(t, x_t), Ax\rangle > 0$ for $\|x\| \geq r$, we obtain

$$\langle Aa, \int_0^T f(t, a)dt\rangle > 0$$

for every $a \in R^n$ such that $\|a\| \geq r$ and hence $QNx \neq 0$ for each $x \in KerL \cap \partial G$ and for each $\lambda \in]0, 1[$, $(1-\lambda)Ac + \lambda QN(c) = 0$ for every $c \in \partial G \cap KerL$. Hence, by the homotopy invariant property of Brouwer degree, we have

$$\begin{aligned} &d([1-\lambda)JA + \lambda JQN] \mid_{KerL},\ G \cap KerL,\ 0) \\ &= d(JQN \mid_{KerL},\ G \cap KerL,\ 0) \\ &= d(JA \mid_{KerL},\ G \cap KerL,\ 0) \\ &= [sgn(detJ')]\ [sgn(detA')] \\ &\neq 0, \end{aligned}$$

Since A is positive definite linear operator, where J', A' are the matrix representation of J and A in same some basis in $KerL$. Thus

$$d(JQN \mid_{KerL},\ G \cap KerL,\ 0) \neq 0$$

Hence the conditions (a), (b) are satisfied and our proof is completed.

Example

$$\dot{x}(t)=ax(t)+bx(t-r)+cx^3(t)+dx^3(t-r)+e(t),$$

where a, b, c, d are constant with $|c|>|d|$ and $e:R\to R$ is continuous and T-periodic, has at least one T-periodic solution.

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