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EINSTEIN EQUATIONS IN NUMERICAL RELATIVITY

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ABSTRACT

Rapid progress in modern computer industries now enables us to solve the Einstein equations numerically. In the first part of this paper we briefly review how to deal with those equations in relativistic astrophysics and cosmology. In the second part we introduce two examples—the Centrella and Wilson's cosmology and the Shapiro and Teukolsky's relativistic stellar cluster.

I. INTRODUCTION

Rapid progress in modern computer industries also changes the world of astrophysics dramatically. Some astrophysical problems which could not be done before are now possible due to the fast calculation of modern computers. Above all, after people found how to solve the Einstein gravitational equations using modern computers, the world of relativistic astrophysics and cosmology changed significantly. This field of astrophysics is called "numerical relativity." It is obvious that the more progress modern computer industries make, the faster numerical relativity grows up. In this paper we briefly introduce the principles of numerical relativity—how to solve the Einstein equations with computers.

Numerical relativity is based on spacetime splitting methods. Among them, Arnowitt, Deser, and Misner (1962, hereafter ADM)'s "3+1"-formalism was recently noticed and developed to solve Einstein equations,

$$G_{\mu\nu}=\kappa T_{\mu\nu}, \tag{1.1}$$

numerically, and it has been widely accepted as a proper way. In equations (1.1) $G_{\mu\nu}$ and $T_{\mu\nu}$ are the Einstein tensor and the stress-energy tensor, respectively, and κ is the constant which is equal to 8π in the units such that $c=G\equiv1$. We set the cosmological constant Λ to be zero in this paper.

In §II we will define several fundamental quantities of numerical relativity, which enables us to establish the "3+1"-spacetime splitting. Smarr and York (1978a, b) will be the main reference of this section. In §III we will discuss the split Einstein equations. York (1979) will be the main reference of this section. In §IV we will discuss two successful examples of numerical relativity—Centrella and Wilson (1983, hereafter CW), and Shapiro and Teukolsky (1985, hereafter ST). We present some final remarks about the stress-energy tensor part in §V.

Throughout this paper $(-+++)$ signs will be used with units such that $c=G=1$. Greek indices will run from 0 to 3 while Latin indices from 1 to 3. A 4-dimensional vector will be expressed as a letter with an arrow.

II. FUNDAMENTAL QUANTITIES OF EACH TIME SLICE

From the point of view of the ADM formalism, the spacetime is a foliation of spacelike hyperspaces connected by timelike curves. Hence the important quantities are $\gamma_{\mu\nu}$, the intrinsic curvature of the hyperspace (the 1st fundamental form), and $K_{\mu\nu}$, the extrinsic curvature of the hyperspace (the 2nd fundamental form). Let $\vec{n}$ be the orthonormal timelike vector of the time slice and $\vec{m}$ be the unit tangent vector of timelike curves. Then $\gamma_{\mu\nu}$ and $K_{\mu\nu}$ are defined by

$$\gamma_{\mu\nu}\equiv g_{\mu\nu}+n_\mu n_\nu \tag{2.1a}$$

and

$$K_{\mu\nu}\equiv -\frac{1}{2}L_{\vec{n}}\gamma_{\mu\nu}, \tag{2.1b}$$

where L means the Lie derivative.

In the ADM formalism the spacetime metric tensor $g_{\mu\nu}$ is set by

$$g_{\mu\nu}=\begin{pmatrix} -\alpha^2+\beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \tag{2.2}$$

where α is the lapse function, Δ(proper time τ)/Δ (coordinate time t), and β^i the shift

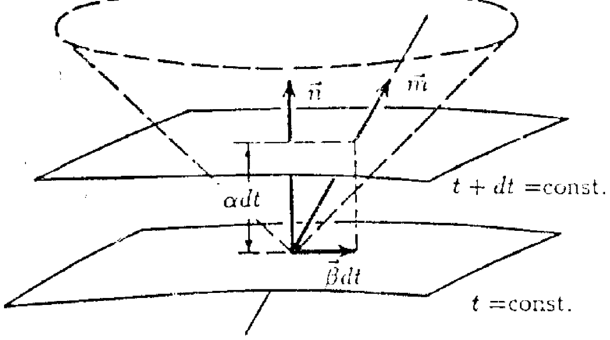

Fig. 1.—Illustrated are parts of two time slices, t=const. and $t+dt$=const., and the relation $\vec{m}=\alpha\vec{n}+\vec{\beta}$. The dashed figure represents a local light cone (adapted from York 1979).

vector with $\beta_i=\gamma_{ij}\beta^i$. Two slices ($t$=const. and $t+dt$=const.) are shown in Figure 1. Here $\vec{n}$ becomes the 4-velocities of the (Eulerian) observer who is at rest in the slice. This observer is moving with respect to the coordinates $\{x^i\}$ due to β^i (Notice that β_i is the coefficient of $dx^i dt$ in the metric [2.2]). The lapse of τ for the observer between two slices ($t=t$ and $t=t+dt$) is given by αdt. Therefore, we get the relation $\vec{m}=\alpha\vec{n}+\vec{\beta}$, where $\beta\equiv(0,\beta^i)$. In two examples in §IV m^μ and n_μ are set equal to $(1,0,0,0)$ and $(-\alpha,0,0,0)$, respectively.

For later use we will introduce the spatial curvature quantities here. We can get the spatial curvature tensor $R^{(3)}_{\mu\nu\lambda\rho}$ from the 4-dimensional spacetime curvature $R_{\mu\nu\lambda\rho}$ through the Gauss-Codazzi relations,

$$R^{(3)}_{\mu\nu\lambda\rho}=\gamma^\alpha_\mu\gamma^\beta_\nu\gamma^\gamma_\lambda\gamma^\delta_\rho R_{\alpha\beta\gamma\delta}-K_{\mu\rho}K_{\nu\lambda}+K_{\mu\lambda}K_{\nu\rho} \tag{2.3a}$$

and

$$D_\alpha K_{\beta\delta}-D_\beta K_{\alpha\delta}=\gamma^\tau_\beta\gamma^\lambda_\alpha\gamma^\mu_\delta n^\nu R_{\tau\lambda\mu\nu}, \tag{2.3b}$$

where D means the spatial covariant derivative, i.e., $D_\lambda\gamma_{\mu\nu}=0$. We contract the spatial curvature tensor to get the spatial Ricci tensor, $R^{(3)}_{\mu\nu}$, and the spatial scalar curvature, $R^{(3)}$, as

$$R^{(3)}_{\mu\nu}=R^{\lambda\,(3)}_{\mu\lambda\nu} \tag{2.4a}$$

and

$$R^{(3)}=\gamma^{\mu\nu}R^{(3)}_{\mu\nu}, \tag{2.4b}$$

respectively.

III. EINSTEIN EQUATIONS

By means of the ADM formalism general relativity becomes a dynamical theory which we view as a Cauchy problem. The Einstein equations (1.1) will be separated into "3+1"-equations as,

$$n^\mu n^\nu G_{\mu\nu}=8\pi n^\mu n^\nu T_{\mu\nu}, \tag{3.1a}$$

$$\gamma^\mu_\lambda n^\nu G_{\mu\nu}=8\pi\gamma^\mu_\lambda n^\nu T_{\mu\nu}, \tag{3.1b}$$

and

$$\gamma^\mu_\lambda\gamma^\nu_\rho G_{\mu\nu}=8\pi\gamma^\mu_\lambda\gamma^\nu_\rho T_{\mu\nu}, \tag{3.1c}$$

by the contractions using n^μ and $\gamma_{\mu\nu}$.

Let us define the Hamiltonian density ρ_H, momentum vector S_λ, and the spatial part of the stress-energy tensor $S_{\lambda\rho}$ to be equal to the right hand sides of equations (3.1),

$$\rho_H \equiv n^\mu n^\nu T_{\mu\nu}, \tag{3.2a}$$

$$S_\lambda \equiv -\gamma^\mu_\lambda n^\nu T_{\mu\nu}, \tag{3.2b}$$

and

$$S_{\lambda\rho} \equiv \gamma^\mu_\lambda \gamma^\nu_\rho T_{\mu\nu}. \tag{3.2c}$$

Contracting the Gauss-Codazzi equations (2.3), we get

$$\gamma^{\alpha\gamma}\gamma^{\beta\delta}R_{\alpha\beta\gamma\delta} = R^{(3)} + K^2 - K_{\mu\nu}K^{\mu\nu}, \tag{3.3a}$$

$$\gamma^\beta_\mu \gamma^\delta_\nu \gamma^{\alpha\gamma} R_{\alpha\beta\gamma\delta} = R^{(3)}_{\mu\nu} + KK_{\mu\nu} - 2K_{\mu\lambda}K^\lambda_\nu, \tag{3.3b}$$

and

$$\gamma^{\alpha\gamma}\gamma^\beta_\mu n^\delta R_{\alpha\beta\gamma\delta} = D^\nu(K_{\mu\nu} - \gamma_{\mu\nu}K), \tag{3.3c}$$

where $R^{(3)}_{\mu\nu}$ and $R^{(3)}$ are the spatial Ricci tensor and the spatial scalar curvature defined in relations (2.4).

Since the left-hand side of (3.3a) is equal to $2n^\mu n^\nu G_{\mu\nu}$, equating this to the right-hand side and using equations (1.1) yields

$$16\pi n^\mu n^\nu T_{\mu\nu} = R^{(3)} + K^2 - K_{\mu\nu}K^{\mu\nu}, \tag{3.4a}$$

and, in a similar way, we get

$$8\pi\gamma^\mu_\lambda n^\nu T_{\mu\nu} = D^\nu(K_{\lambda\nu} - \gamma_{\lambda\nu}K), \tag{3.4b}$$

from equations (3.3c). Substituting definitions (3.2) into equations (3.4), we get 1 Hamiltonian constraint equation and 3 momentum constraint equations,

$$R^{(3)} + K^2 - K_{\mu\nu}K^{\mu\nu} = 16\pi\rho_H \tag{3.5a}$$

and

$$D^\nu(K_{\mu\nu} - \gamma_{\mu\nu}K) = 8\pi S_\mu. \tag{3.5b}$$

Notice also the relations,

$$\gamma^\alpha_\mu \gamma^\gamma_\nu n^\beta n^\delta R_{\alpha\beta\gamma\delta} = \gamma^\alpha_\mu \gamma^\gamma_\nu \gamma^{\beta\delta} R_{\alpha\beta\gamma\delta} - \gamma^\alpha_\mu \gamma^\gamma_\nu n^\beta n^\delta R_{\alpha\gamma}, \tag{3.6a}$$

which give the 6 evolution equations

$$L_{\vec{m}}K_{\mu\nu} = -D_\mu D_\nu \alpha + \alpha(R^{(3)}_{\mu\nu} + KK_{\mu\nu} - 2K_{\mu\lambda}K^\lambda_\nu) - 8\pi\alpha\left(S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}S\right)$$
$$-4\pi\alpha\rho_H\gamma_{\mu\nu} + L_{\vec{\beta}}K_{\mu\nu}, \tag{3.6b}$$

where S is the trace of $S_{\mu\nu}$. In practice, the trace of equations (3.6b),

$$L_{\vec{m}}K = -D_\lambda D^\lambda \alpha + \alpha\left\{K^{\mu\nu}K_{\mu\nu} + \frac{1}{2}(\rho_H + S)\right\} + L_{\vec{\beta}}K, \tag{3.7}$$

can be used if the astrophysical problem provides enough symmetry. Equation (3.7) is called "the lapse equation."

IV. EINSTEIN EQUATIONS IN TWO EXAMPLES

1. PLANAR NUMERICAL COSMOLOGY

CW construct a planar cosmological code. They used a time coordinate t and three spatial Cartesian coordinates, x, y and z. Their metric is set by(CW, eq. [2.1] and CW, eq. [2.2])

$$g_{\mu\nu}=\begin{pmatrix} -\alpha^2+\beta_i\beta^i & \beta_x & 0 & \beta_z \\ \beta_x & A^2 & 0 & 0 \\ 0 & 0 & A^2h^2 & 0 \\ \beta_z & 0 & 0 & A^2 \end{pmatrix}, \tag{4.1}$$

where all quantities are functions of z and t only. The conformal factor A is a generalization of the cosmological scale factor and h is an anisotropic shear variable. K^μ_ν are set by (CW, eq. [2.3]),

$$K_\mu{}^\nu=\begin{pmatrix} K^x_x & 0 & K^z_x \\ 0 & K^y_y & 0 \\ K^z_x & 0 & K^z_z \end{pmatrix}. \tag{4.2}$$

They employed the gauge condition $K=K(t)$ (CW, eq. [2.4]), which is often called "constant-mean-curvature" slicing. This, together with $K_1 \equiv K^x_x - K^y_y$ (CW, eq. [2.5]), will evolve K^μ_ν.

The spactime metric (2.8) now enable us to express the constrain equations as follows (CW, eq. [4.4], CW, eq.[4.5], and CW, eq. [4.6]),

$$\partial_z\partial_z h+2\frac{\partial_z A}{A}\partial_z h=h\left\{-2\frac{\partial_z\partial_z A}{A}+\left(\frac{\partial_z A}{A}\right)^2-\frac{A^2}{2}(K^j_i K^i_j-K^2+2\rho_H)\right\}, \tag{4.3a}$$

$$\partial_z K^z_z+(3K^z_z-K)\left(\frac{\partial_z A}{A}+\frac{1}{2}\ \frac{\partial_z h}{h}\right)+\frac{1}{2}\ \frac{\partial_z h}{h}K_1=S_z, \tag{4.3b}$$

and

$$\partial_z K^z_x+\left(3\frac{\partial_z A}{A}+\frac{\partial_z h}{h}\right)K^z_x=S_x. \tag{4.3c}$$

Notice that CW use $\kappa=1$, and employ symbols such as(CW, eq. [3.7], CW, eq. [3.8], and CW, eq. [3.9])

$$d\equiv\sigma w^2-P=(\text{our } \rho_H), \tag{4.4a}$$

$$j_i=-\sigma w^2 V_i=-(\text{our } S_i), \tag{4.4b}$$

and,

$$S_{ij}=P\gamma_{ij}+\sigma w^2 V_i V_j. \tag{4.4c}$$

The definitions of σ, w, P, and V_i will be clear in §V.

The evolution equations(CW, eq. [3.14]) become (CW, eq. [4.8]),

$$\partial_t K_1=\beta^z\partial_z K_1+\alpha K K_1-2\alpha(K^z_x)^2+\frac{1}{A^2 h}\left\{\partial_z\alpha\partial_z h+\alpha\left(\partial_z\partial_z h+\frac{\partial_z A}{A}\partial_z h\right)\right\}$$
$$-\frac{\alpha}{A^2}\frac{S^2{}_x}{\sigma w^2}, \tag{4.5a}$$

and the lapse equation turns out to be(CW, eq. [4.1]),

$$\partial_z\partial_z\alpha+\left(\frac{\partial_z A}{A}+\frac{\partial_z h}{h}\right)\partial_z\alpha-\alpha A^2\left\{K^i_j K^j_i+\sigma\left(w^2-\frac{1}{2}\right)+P\right\}+A^2\partial_t K=0. \tag{4.5b}$$

We can get some auxiliary equations from the definition of $K_{\mu\nu}$, which are (CW, eq. [3.13])

$$\partial_t\gamma_{ij}=-2\alpha\gamma_{ik}K^k_j+\gamma_{jk}D_i\beta^k+\gamma_{ik}D_j\beta^k. \tag{4.6}$$

Substituting each component in equation (4.6), we get(CW, eq. [4.2], CW, eq. [4.3], CW, eq. [4.7], and CW, eq. [4.9]),

$$\partial_z\beta^z=\frac{1}{2}\alpha(3K^z_z-K-K_1), \tag{4.7α}$$

$$\partial_z\beta^x=2\alpha K^z_x, \tag{4.7b}$$

$$\partial_t A=\beta^z\partial_z A-\frac{1}{2}\alpha A(K+K_1-K^z_z), \tag{.7c}$$

and

$$\partial_t h=\beta^z\partial_z h+\alpha h K_1. \tag{4.7d}$$

2. RELATIVISTIC STELLAR DYNAMICS

ST construct a relativistic stellar dynamics code. This code corresponds to the possible origin of QSOs via the collapse of dense star clusters to supermassive black holes. They used a time coordinate t and three spherically smmetric coordinates, r, θ, and φ. Their metric is set by (ST, eq. [1]),

$$g_{\mu\nu}=\begin{pmatrix} -\alpha^2+\beta_r\beta^r & \beta_r & 0 & 0 \\ \beta_r & A^2 & 0 & 0 \\ 0 & 0 & A^2r^2 & 0 \\ 0 & 0 & 0 & A^2r^2\sin^2\theta \end{pmatrix}, \tag{4.8}$$

where all quantities are functions of r and t only. Hereafter β means the only shift vector component β^r.

For the extrinsic curvature, they chose "maximal time-slicing", $K=0$. This with the condition of isotropy reduces the degrees of freedom of $K_{\mu\nu}$ to one because $K^\theta_\theta=K^\varphi_\varphi$ and $K^r_r=-2K^\theta_\theta$. From the definitions of $K_{\mu\nu}$, after some calculations, we can get shift equation and metric evolution equation as(ST, eq. [22] and ST, eq. [23])

$$\beta = -\frac{3r}{2}\int_r^\infty \frac{\alpha K^r{}_r}{r} dr \tag{4.9a}$$

and

$$\partial_t A = \beta\left(\partial_r A + \frac{A}{r}\right) + \frac{1}{2}\alpha A K^r_r. \tag{4.9b}$$

The spacetime metric (4.8) now enables us to express the constrain equations (3.6) as (ST, eq. [29] and ST, eq. [30])

$$\frac{1}{r^2}\partial_r(r^2\partial_r A^{1/2}) = -\frac{1}{4}A^{5/2}\left(8\pi\rho_H + \frac{3}{4}(K^r_r)^2\right) \tag{4.10a}$$

and

$$K^r_r = \frac{8\pi}{A^3 r^3}\int_0^r A^3 r^3 S_r dr. \tag{4.10b}$$

Notice that they use

$$\rho = (\text{our } \rho_H), \tag{4.11a}$$

$$t_i = -(\text{our } S_i), \tag{4.11b}$$

and

$$K = K^r_r \neq (\text{our } K). \tag{4.11c}$$

The lapse equation (3.7) has the form(ST, eq. [31]),

$$\rho_r(Ar^2\partial_r\alpha) = \frac{1}{2}\alpha A^3 r^2\{8\pi(\rho_H + S) + 3(K^r_r)^2\}, \tag{4.12}$$

in this case.

V. CONSERVATION EQUATIONS

Finally we will review briefly how numerical relativists deal with the right-hand side of equations (1.1).

The stress-energy tensor of the fluid, $T^{(\text{fluid})}_{\mu\nu}$ for a Lagrangian observer whose 4-velocity is given by a timelike unit vector $\vec{u}$ is,

$$T^{(\text{fluid})}_{\mu\nu} = \mu u_\mu u_\nu + 2q_{(\mu}u_{\nu)} + ph_{\mu\nu} + \Pi_{\mu\nu}, \tag{5.1}$$

where μ is the energy density, q^μ is the energy flux vector, p is the isotropic pressure, $h_{\mu\nu}$ is the projection tensor,

$$h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu, \tag{5.2}$$

and $\Pi_{\mu\nu}$ is the anisotropic pressure. Here notice that $\vec{u}$ is not equal to $\vec{m}$ or $\vec{n}$ generally. In equations (5. 1) μ, q_μ, and $\Pi_{\mu\nu}$ are defined by,

$$\mu \equiv \rho(1+\epsilon), \tag{5. 3a}$$

$$q_\mu \equiv -kTh_\mu^\nu[a_\nu+\partial_\nu(\ln\ T)], \tag{5. 3b}$$

and,

$$\Pi_{\mu\nu} \equiv -\lambda\sigma_{\mu\nu}, \tag{5. 3c}$$

where ρ is the rest energy density, ϵ is the specific internal energy density, k is heat conduction, T is temperature, λ is viscosity coefficient, and $\sigma_{\mu\nu}$ is the shear tensor.

Quantities $\rho_H^{(\text{fluid})}$, $S_\mu^{(\text{fluid})}$, and $S^{(\text{fluid})}$ will be set through (3. 2). For simplicity, $T_{\mu\nu}^{(\text{fluid})}$ is assumed to be that of thə perfect fluid in many astrophysical problems, which has the form,

$$T_{\mu\nu}^{(\text{fluid})} = \rho h u_\mu u_\nu + P g_{\mu\nu}, \tag{5. 4}$$

The particle conservation equation, the energy conservation equation, and the momentum conservation equations are, respectively,

$$\nabla_\lambda(\rho u^\lambda)=0, \tag{5. 5a}$$

$$u^\mu \nabla^\nu T_{\mu\nu}^{(\text{flud})}=0, \tag{5. 5b}$$

and,

$$h_\lambda^\mu \nabla^\nu T_{\mu\nu}^{(\text{fluid})}=0, \tag{5. 5c}$$

where ∇ means the spatial covariant derivative, i.e., $\nabla_\lambda g_{\mu\nu}=0$.

Now we define the following variables,

$$u \equiv -\vec{n}\cdot\vec{u} = \alpha u^t, \tag{5. 6a}$$

$$D \equiv \rho u, \tag{5. 6b}$$

$$E \equiv \rho\epsilon u, \tag{5. 6c}$$

and,

$$v^i \equiv \frac{u^i}{u^t}, \tag{5. 6d}$$

and, substituting (5. 4) into (2. 5b), rewrite the momentum vector as,

$$S_i = \rho h u u_i = (D+E+Pu)u_i. \tag{5. 6e}$$

Then the conservation equations (5. 5) yield the following equations which will be used for the time evolution of D, E, and S_i,

$$\partial_t(D\sqrt{\gamma})+\partial_i(Dv^i\sqrt{\gamma})=0, \tag{5. 7a}$$

$$\partial_t(E\sqrt{\gamma})+\partial_i(Ev^i\sqrt{\gamma})=-P[\partial_t(u\sqrt{\gamma})+\partial_i(uv^i\sqrt{\gamma})], \tag{5. 7b}$$

and,

$$\partial_t(S_j\sqrt{\gamma})+\partial_i(S_jv^i\sqrt{\gamma})=-\alpha\sqrt{\gamma}\Big(\partial_jP+\frac{1}{2}\rho hu_\mu u_\nu\partial_jg_{\mu\nu}\Big), \qquad (5.7c)$$

where γ is the trace of $\gamma_{\mu\nu}$. Notice that CW use w=(our u), σ=(our ρh), and V_i=(our v_i)/α, and ST do not use equations (5.1).

Equations (5.7) are used for the first time by Wilson in his pioneering work (Wilson 1971). They are also used by Hawley, Smarr, and Wilson (1984) to develop an axisymmetric code to study hydrodynamic accretion flows in a fixed black hole gravitational field.

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