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Analysis of A Two-Machine One Repairman Problem

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Abstract

This paper combines research in the areas of replacement and machine interference. Although over the past three decades there has been a great deal of research in the area of optimal replacement for stochastically deteriorating equipment and research dealing with machine interference problems; there has been a lack of research when these two areas are combined. However, the melding of these two well-known areas yields a very practical problem which demands theoretical investigation.

In this paper we derive the steady state probabilities with a control limit policy for a two-machine one repairman problem. The control policy is a simple age dependent control described by the control limit, t*. Once t* is fixed, the steady state probabilites that one, two, and no machines are working will be obtained.

1. Background

Mathematical sophistication of replacement models has increased in parallel to the growth in the complexity of modern systems. These mathematical developments have drawn the attention of many researchers who have published a host of research in the areas of optimal

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replacement and preventive maintenance. Research in the machine interference problem has taken place concurrently with the research in optimal replacement, but little attention has been paid to the problem of optimal replacement in a machine interference setting. (Machine interference arises when a fixed set of machines is under the care of a limited number of repairman. If more machines need repair than the number of repairman available, the failed machines form a queue.)

Thus, this paper opens up a new class of problems for investigation by combining research in the areas of the replacement and machine interference. The melding of these two well-known areas yields a very practical problem which demands theoretical investigation. We show in this paper how to obtain the steady-state probabilities for a two-machine one repairman problem.

One of the original replacement problems was determination of an optimal age-replacement policy for a machine subject to failure (Ackoff and Sasieni, 1968). The problem was to determine the optimal replacement age of a machine when the two major cost considerations were the replacement cost incurred during a plannned replacement and the larger cost incurred if replacement was made due to machine failure. In some industrial settings there are a limited number of repairman available so that, in practice, decisions to replace major process equipment are made simply because repairman are idle. Such a management decision is partially based on the desire to avoid situations where the equipment to be replaced is queued and thus remains idle. It is often felt (intuitively) by management that it is better to replace equipment early than to take a chance on equipment being idle because of the occurance of more failures than there are repairman to handle the work load. However, specific quantitative guidelines for such decisions are not presently available. (A complete survey of replacement models is contained in Valdez-Flores and Feldman (1988); however, none of the available literature deals with replacement in a machine interherence context)

Repair or replacement of equipment when a limited number of repairmen are available is commonly called the machine interference problem, namely, a G/M/1 system with a finite calling population. Bunday and Scraton (1980) derives the steady-state probabilities for the finite population G/M/r system and show that they are identical to the finite population M/M/r system; thus, much of the work completed for the Markov machine interference problem would seem to apply to the general failure distribution case. The problem of allocating machines to repairmen has been extensively studied; Palm (1958) was among the first to consider this problem. Jaiswal and Thiruvengadam (1963) and Elsayed (1981) consider two repair policies for machine interference problems with two failure modes and repair times. Carpenito and White (1976) extend interference problems to non-identical machines and non-identical repairmen. These above mentioned models are designed to determine the appropriate number of machines to assign to a repairman; however, the problem of determining the optimal age-replacement time within the machine interference context has not be investigated. Furthermore, once an age-based control limit is attached to the machine interference problem, the work with an exponential failure law is no longer relevant.

The machine interference problem can also be viewed as a closed queueing network in which a $\cdot/G/\infty$ system feeds into a $\cdot/M/1$ system which in turn feeds back into the first system. Since this network gives rise to a reversible process, the steady-state probabilities for the number of machines in each system is identical to the steady-state probabilities of the system formed when the general faliure distribution is replaced by an exponential distribution with the same mean time to failure (Kelly 1979). Difficulty again arises because of our desire to impose control on the process. Specifically, the reversibility property is lost when an age-dependent control policy is imposed on the closed $\cdot/G/\infty \leftrightarrow \cdot/M/1$ network; therefore, analyses on a $\cdot/M/\infty \leftrightarrow \cdot/M/1$ are not directly applicable.

In this paper, we show how the derivation by Bundy and Scraton (1980) can be modified to obtain the steady-state probabilities of the number of machines working under a control limit policy.

2. The Problem

Consider a system consisting of two identical and independent machines under the care of one repairman. The time to failure of each machine is a random variable with distribution function G. The time to replace (or totally repair) a machine is exponentially distributed with mean time $1/\mu$. The control structure to be imposed on this system is a simple age-dependent control policy described by the control limit t^* where the control limit is only operational if the repairman is idle. That is, if both machines are working, then as soon as a machhine ages past the limit t^* that machine is taken out of service for replacement. If the repairman is busy, then the control limit is ignored and the only time a machine is sent to the tepairman is upon failure. (We assume that the random time to replace the machine has the same in distribution for failed machines and for those taken out of service early.)

3. Steady-State Probabilities

Let G be the disribution of failure times for a machine and let $\overline{G}=1-G$. Our derivation follows that of Bunday and Scraton (1980) and therefore we assume that the failure time is a continuous random variable with the density function of failure times being given by g. Because the derivation of steady-state probabilities depends on the continuous assumption, we shall approach the control limit problem as a limiting process. That is for a small ε greater than zero, define the following function :

$$\tilde{G}_\varepsilon(t) = \begin{cases} \bar{G}(t) & \text{for } t < t^*, \\ \psi(t) & \text{for } t^* \leq t < t^* + \varepsilon, \\ 0 & \text{for } t \geq t^* + \varepsilon. \end{cases}$$

where ψ is a (steeply decreasing) function such that $\tilde{G}_\varepsilon$ is continuous and once differentiable. Thus, $\tilde{G}_\varepsilon$ limits to the complement of the failure time distribution under the control limit t^* as ε approaches zero. Finally, let $\tilde{g}_\varepsilon$ denote the negative of the derivative of $\tilde{G}_\varepsilon$

The steady-state probability that both machines are operating and the age of one machine is in the interval (t_1, t_1+dt_1) and the age of the second machine is in the interval (t_2, t_2+dt_2) is given by $Q_2(t_1, t_2)\,dt_1 dt_2$. The steady-state probability that one particular machine is operating and the age of that machine is in the interval (t_1, t_1+dt_1) is given by $Q_1(t_1)\,dt_1$. Finally, the steady-state probability that no machines are operating is given by Q_0. (The functions Q_2, Q_1, and Q_0 are dependent on ε but we have not included the ε in the subscript for ease of notation).

The following equations, which are modifications of Bunday and Scraton's equations (2.8) – (2.13), are intuitive given that μ is the repair rate, $g(t)/\bar{G}(t)$ is the failure rate for a machine of age t when only one machine is operating, and $\tilde{g}(t)/\tilde{G}(t)$ is the failure rate per machine of age t when both machines are operating. The system of equations that define the steady-state probabilities are :

$$\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)Q_2(t_1,t_2) = -Q_2(t_1,t_2)\left[\frac{\tilde{g}_\varepsilon(t_1)}{\tilde{G}_\varepsilon(t_1)}+\frac{\tilde{g}_\varepsilon(t_2)}{\tilde{G}_\varepsilon(t_2)}\right] \text{ for } 0 < t_1,t_2 < t^* + \varepsilon, \tag{1}$$

$$\frac{\partial}{\partial t_1}Q_1(t_1) = -Q_1(t_1)\left[\frac{g_1(t_1)}{\tilde{G}_1(t_1)}+\mu\right]+\int_0^{t^*+\varepsilon}Q_2(t_1,s)\frac{\tilde{g}_\varepsilon(s)}{\tilde{G}_\varepsilon(s)}ds \quad \text{for } t_1 > 0, \tag{2}$$

$$0 = \mu Q_0 - 2\int_0^\infty Q_1(s)\frac{g(s)}{\bar{G}(s)}ds \tag{3}$$

$$Q_2(t_1,0) = \begin{cases}\mu Q_1(t_1) & \text{if } t_1 < t^* + \varepsilon \\ 0 & \text{if } t_1 \geq t^* + \varepsilon\end{cases} \tag{4}$$

$$Q_1(0) = \frac{1}{2}\mu Q_0 + \mu\int_{t^*+\varepsilon}^\infty Q_1(s)ds. \tag{5}$$

There are two major differences between our equations and the equations of Bunday and Scraton. First, Since $\tilde{G}_\varepsilon$ is the relevant function when two machines are working, Equations (1) and (2) involve the function $\tilde{G}_\varepsilon$ instedad of $\bar{G}$. Second, if only one machine is working and that machine is older than $t^*+\varepsilon$, then a repair will cause the older machine to be immediately sent to the repairman; thus Eq. (5) has a term representing that possibility.

To solve this system of equations, we shall rewrite them in terms of R_2, R_1 and R_0 defined by

$$\begin{aligned} Q_2(t_1,t_2) &= R_2(t_1,t_2)\tilde{G}_\varepsilon(t_1)\tilde{G}_\varepsilon(t_2) \\ Q_1(t_1) &= R_1(t_1)\bar{G}(t_1) \\ Q_0 &= R_0. \end{aligned} \tag{6}$$

Equations (1) – (5) are now expressed in terms of R_2, $R_1$and R_0 to become;

$$\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)R_2(t_1,t_2) = 0 \text{ for } t_1,t_2 < t^* + \varepsilon, \tag{7}$$

$$\frac{\partial}{\partial t_1} R_1(t_1) = -\mu R_1(t_1) + \int_0^{t^*+\varepsilon} R_2(t_1, s) \frac{\tilde{G}_\varepsilon(t_1)}{\bar{G}_\varepsilon(t_1)} \tilde{g}_\varepsilon(s) ds \quad \text{for} \quad t_1 > 0, \tag{8}$$

$$0 = \mu R_0 - 2\int_0^\infty R_1(s) g(s)\, ds \tag{9}$$

$$R_2(t_1, 0) = \begin{cases} \mu R_1(t_1) & \text{if} \quad t_1 < t^* \\ \mu R_1(t_1) \bar{G}(t_1) / \tilde{G}_\varepsilon(t_1) & \text{if} \quad t^* < t_1 < t^* + \varepsilon \\ 0 & \text{if} \quad t_1 \geq t^* + \varepsilon \end{cases} \tag{10}$$

$$R_1(0) = \frac{1}{2} \mu R_0 + \mu \int_{t^*+\varepsilon}^\infty R_1(s) \bar{G}(s) ds. \tag{11}$$

To solve these equations, we first note that a constant is the only solution to Eq. (7); thus,

$$R_2(t_1, t_2) = \begin{cases} k & \text{if} \quad t_1, t_2 < t^* + \varepsilon \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

where k is to be determined by boundary conditions (we shall eventually use the norming equation to obtain k).

We use Eq. (12) and the facts that $\int_0^{t^*-\varepsilon} \tilde{g}_\varepsilon(s) ds = 1$ and $\tilde{G}_\varepsilon(t) = \bar{G}(t)$ for $t < t^*$ to obtain

$$\frac{\partial}{\partial t_1} R_1(t_1) = \begin{cases} -\mu R_1(t_1) + k & \text{if} \quad t_1 \leq t^*, \\ -\mu R_1(t_1) + k \frac{\tilde{G}_\varepsilon(t_1)}{\bar{G}(t_1)} & \text{if} \quad t^* < t_1 < t^* + \varepsilon, \\ -\mu R_1(t_1) & \text{if} \quad t_1 \geq t^* + \varepsilon. \end{cases}$$

The top and bottom branches of this equation are easily soved to obtain

$$R_1(t_1) = \begin{cases} \frac{1}{\mu} k & \text{if} \quad t_1 \leq t^*, \\ R_1(t_1) & \text{if} \quad t^* < t_1 < t^* + \varepsilon, \\ k_1 e^{-\mu t_1} & \text{if} \quad t_1 \geq t^* + \varepsilon. \end{cases} \tag{13}$$

where k_1 is a constant.

Wh substitute Eqs. (12) and (13) into Eqs. (9) and (11) to obtain

$$R_0 = \frac{2}{\mu^2} \left(kG(t^*) + \int_{t^*}^{t^*+\varepsilon} R_1(s) g(s) ds + k_1 \int_{t^*+\varepsilon}^\infty \mu_1 e^{-\mu s} g(s) ds \right)$$

$$R_0 = \frac{2}{\mu^2} k - 2k_1 \int_{t^*+\varepsilon}^\infty e^{-\mu s} \bar{G}(s) dts. \tag{14}$$

By setting these two equations equal to each other, we obtain the value of the constant

k_1 as

$$k_1 = \frac{1}{\mu} k e^{-\mu t^*}. \tag{15}$$

(In solving for k_1, the expression $\overline{G}(t) = \int_t^{\infty} g(s)\,ds$ is substitued into Eq. (14) and then the order of integration is reversed.)

The basic probabilities are now obtained by combining Eqs. (6) - (15) and taking the limit as ε approaches zero. This yields

$$Q_2(t_1, t_2) = \begin{cases} k\bar{G}(t_1)\bar{G}(t_2) & \text{if} \quad t_1, t_2 < t^* \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

$$Q_1(t_1) = \begin{cases} k\bar{G}(t_1)/\mu & \text{if} \quad t_1 \leq t^*, \\ k e^{-\mu(t_1 - t^*)}\bar{G}(t_1)/\mu & \text{if} \quad t_1 > t^*, \end{cases} \tag{17}$$

$$Q_0 = 2k\left(1 - \mu\int_{t^*}^{\infty} e^{-\mu(s-t^*)}\bar{G}(s)\,ds\right) / \mu^2. \tag{18}$$

The final steady-state probabilities for the number of machines working are obtained by integrating the above expressions over time. The final step is to observe that $Q_1(t_1)$ is the probability associated with a particular machine and since we do not care which machine is working we have $q_1 = 2\int_0^{\infty} Q_1(s)\,ds$. Before giving the final equations, we simplify the expressions by defining two quantities in terms of the control limit, t^*, as

$$m_{t^*} = \int_0^{t^*} \bar{G}(s)\,ds$$

$$\Gamma_{t^*} = \int_{t^*}^{\infty} e^{-\mu(s-t^*)}\bar{G}(s)\,ds$$

Thus, m_{t^*} is the mean time until failure for a machine operating under the control limit and Γ_{t^*} is similar to a (translated) Laplace transform. Therefore, the probabilities that 2, 1, or 0 machines are working is given by

$$q_2 = k m_{t^*}^2 \tag{19}$$

$$q_1 = 2k[m_{t^*}^2 + \Gamma_{t^*}]/\mu^2 \tag{20}$$

$$q_0 = 2k[1 - \mu\Gamma_{t^*}]/\mu^2, \tag{21}$$

where k is determined so that $q_0 + q_t + q_L = 1$.

4. CONCLUSIONS

This paper analyzed age replacement polices for a stochastically deteriorating equipment within the context of a machine interference setting. Most replacement models reported in the literature considered "*to replace or not to replace*" according to age of a machine only so that the expected cost of replacement per unit time could be minimized. Once a decision is made to replace, immediate replacement was assumed. However, previous research on machine interference problems concentrated on assinging the appropriate number of machine to a repairman or on finding the optimal number of repairman to numinize the system cost for a fixed number of machines. In machine interference problems, the age replacement policy has never been considered. The model developed in this paper combined these two research areas. Although our attention was restricted to the two-machine one repairman problem, we expect this work to stimulate further research related to the two areas.

Under the assumptions that the time to failure of each machine has a general distribution **G** and the repair (or replacement) times exponentially distributed with rate μ, the steady-state probabilities were derived for a given control policy t^*.

This work opens up a new research area which considers traditional age replacement problems within the context of machine interference setting. However, our work was restricted to the two-machine one repairman problem. Thus, a possible extension to this paper is to extend these results to an n-machine r repairmen problem for general failure time distributions. Further research could also consider state-replacement policies instead of age-replacement policies.

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