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APPLICABILITY OF THE LANCHESTER MODEL TO THE MANY-ON-MAMY DIRECT-FIRE ENGAGEMENT

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Abstract

The Lanchester equations are widely used for modelling the direct-fire land battle. However, it is recognized that the Lanchester based models are less applicable to direct-fire land combat when the battle size is small, the forces are near parity or the inter-firing times of the combatants do not follw a negative exponential distribution. A comprehensive investigation has been conducted to establish the circumstances under which the Lanchester based models are applicable.

1. INTRODUCTION

A model is a simplified representation of the real world or that part of the world of interest. It is potentially useful for analysts and decision makers to study their problems by using models, but the concept of model is very broad depending on the subject.

In the military field, even though many problems can not be solved and answers can only be determined through war, a combat model is extensively used today for defence planning

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studies. For example, the combat model can be used for making choices and also comparing alternative weapon systems, tactics, etc. Thus, military analysts are seeking for 'better' models to obtain 'better' answers by reflecting more realistic battle situations.

The first contribution to the mathematics of conflict was made by the British engineer Frederick William Lanchester during the World War I.

1. 1 Deterministic Lanchester Models

Lanchester became interested in the military use of aircraft around 1914, and in 1916, he published his work in a book[10] in which he described the well-known 'Square Law' for direct fire combat, and the 'Linear Law' for indirect fire, as a pair of coupled differential equations.

Brackney[2] introduced a 'Mixed Law' for the situation in which one side attacks and the second defends a fixed area. The attacker's attrition rate follows the Square Law and the defender's attrition rate follows the Linear Law. Deitchman[7] applied the Mixed Law to guerrilla warfare in which the ambushers fire 'aimed-fire' and the ambushed fire 'area-fire'.

Peterson[12] formulated a 'Logarithmic Law', which characterizes the early stages of a small-unit engagement where the vulnerability of a force dominates its ability to acquire enemy targets.

Owen[11] developed a 'General Law'. In this law, when visibility is poor, the number of combatants is small or the rate of fire is very rapid, the General Law reduces to the Linear Law. When visibility is good, the number of combatants is large, or the rate of fire is very low, the General Law reduces to the Square Law.

1. 2 Exponential Lanchester Models

Lanchester represented the strength of the surviving forces with continuous functions having continuous derivatives with respect to time in a battle. However, the state of a combat process at a gven point in time is usually specified by the number of survivors for each force which are random variables as pointed out by Clark[4]. Such a model will be referred to as an Exponential Lanchester model when one of the underlying assumptions is that the inter-firing times follow a negative exponential distribution.

The early work on the stochastic formulation was done by Koopman[9] during the World War Ⅱ. He reformulated the Lanchester equations in a stochastic form. Snow[13] also developed a stochastic analysis of the Lanchester Square Law model. Gye and Lewis[8] developed an analytic expression for the terminal distribution of the number of survivors and they also suggested an approximate formula using a normal distribbution. The approximate formula closely approximated the exact solution as the initial number of combatants increased. Weale[14] suggested a method for the computation of the probability distribution of the battle state at a given time. Daly and Hagues[6] developed an alternative simulation technique for the solution of the Exponential Lanchester model.

1.3 Discussion And Aim

A considerable amount of research has also been done in comparing the Deterministic Lanchester model('DL' model) and the Exponential Lanchester model('EL' model).

The Deterministic Lanchester model is often referred to as the mean of the Exponential Lanchester models but Clark[4] showed that the 'DL' model is not necessarily the mean of the 'EL' model and derived the error term.

Craig[5] examined the fundamental differences between the Deterministic and the Exponential Lanchester models. He derived the function of the differences dependent on time and then concluded that if the forces are not near parity and the initial number of combatants on both sides are relatively "large", then a Deterministic Lanchester model can adequately represent an Exponential Lanchester model.

Gye and Lewis[8] showed that the Deterministic Lanchester model gives a reasonable representation of the Exponential Lanchester model when one side is dominant, but that they are quite different if the forces are of roughly equal strength.

Ancker and Gafarian[1] re-investigated Craig's results and concluded that both the Exponential Lancheter model and the Deterministic Lanchester model are biased at any combat time t (or any range of times) and that the Deterministic Lanchester model should not be used to give universal bounds for the Exponential Lanchester model.

As discussed above, the Deterministic Lanchester model is based on a pair of coupled differential equations representing a many-on-many direct-fire combat situation, but when small number of weapons are involved, the underlying assumptions are less easy to accept.

The Exponential Lanchester model is the stochastic version of the Deterministic Lanchester model and in the many-on-many direct-fire combat situation represents the attrition to each side as a Poisson process in which the inter-firing times are therefore implicitly taken as having a negative exponential distribution. If the inter-firing times are not negative exponential, the Exponential Lanchester model may give a misleading result.

Thus in this paper, an exhaustive evaluation of the applicability of the widely used Deterministic and Exponential Lanchester models to a many-on-many direct-fire engagement on land is conducted.

2. LANCHESTER EQUATION

2. 1 Deterministic Lanchester Equation

Deterministic Lanchester models are based on the assumpion that the attrition suffered by either side in battle is a function of the numerical strengths of the forces involved and the efficiency of their respective weapons. The Lanchester formulation for the direct-fire battle is known as Lanchester's Square Law.

Lanchester made the following assumptions for the Square Law.

1. Two forces attack each other. Each combatant on each side is within weapon range of all combatants of the other side.
2. Combatants on each side are identical but the combatant on one side may have a different kill rate to the opposing combatant.
3. Each firing combatant is sufficiently well aware of the location and condition of all enemy combatants so that when a target is killed, fire may immediately be switched to a new target.
4. Fire is uniformly distributed over surviving targets.

The Lanchester equations for the Square Law are then :

$$\frac{db}{dt} = -\rho r$$

$$\frac{dr}{dt} = -\beta b$$

where,

b is number of Blue combatants at time t

r is number of Red combatants at time t

β is kill rate of each Blue

ρ is kill rate of each Red

The solution of the Lanchester equation, with time eliminated, is

$$\beta(B_0^2-b^2)=\rho(R_0^2-r^2)$$

where,

B_0 is initial number of the Blue side

R_0 is initial number of the Red side

and the time-dependent solution of the Lanchester equation is

$$b(t) = B_0 \cosh \sqrt{\beta\rho}t - R_0\sqrt{\frac{\rho}{\beta}} \sinh \sqrt{\beta\rho}t$$

$$r(t) = R_0 \cosh \sqrt{\beta\rho}t - B_0\sqrt{\frac{\beta}{\rho}} \sinh \sqrt{\beta\rho}t$$

which are valid in the interval $(0, t_f)$, where t_f is time to annihilation.

$$t_f = \begin{cases} 1/\sqrt{\beta\rho}\tanh^{-1}(\sqrt{\rho/\beta}\frac{R_0}{B_0}) & \text{if } \frac{\beta B_0^2}{\rho R_0^2} > 1 \\ 1/\sqrt{\beta\rho}\tanh^{-1}(\sqrt{\beta/\rho}\frac{B_0}{R_0}) & \text{if } \frac{\beta B_0^2}{\rho R_0^2} < 1 \\ \infty & \text{if } \frac{\beta B_0^2}{\rho R_0^2} = 1 \end{cases}$$

The power ratio is defined as $\beta B_0^2/\rho R_0^2$

While, Lanchester's Linear Law model is for the indirect-fire battle. The model has, there fore, different form from that of the Square Law.

2.2 Exponential Lanchester Equation

In a Deterministic Lanchester model, the course of a battle has no random property. However, an equivalent stochastic formulation of the Lanchester concept was developed to take some account of the random nature of combat.

The assumptions go with those of the Square Law model except that each combatant kills opposing combatants randomly at its fixed kill rate.

Consider $P(b, r, t)$, the probability that numbers of surviving combatants of the Blue and Red sides are b and r respectively at time t after the start of battle.

Then the probability that Blue (size b) will obtain a kill in time $\triangle t$ is $\beta b \triangle t$, and the probability likewise for Red is $\rho r \triangle t$. Thus the probability of being in the state (b, r) at time $t + \triangle t$ is expressed by

$$P(b, r, t+\triangle t) = P(b, r, t)\,(1-\beta b\triangle t)\,(1-\rho r\triangle t) + P(b, r+1, t)\,\beta b \triangle t(1-\rho r\triangle t)$$
$$+ P(b+1, r, t)\,\rho r\triangle t\,(1-\beta b\triangle t) + 0(\triangle t)$$

where,

$0(\triangle t)$ is the probability of more than one casualty occurring in a time interval $\triangle t$ and a function such that

$$\lim_{\triangle t \to 0} \frac{0(\triangle t)}{\triangle t} = 0$$

After manipulating terms and dividing by $\triangle t$, we then obtain the differential equation by taking the limit as $\triangle t \to 0$,

$$\frac{dP(b, r, t)}{dt} = -(\beta b + \rho r)P(b, r, t) + \beta b P(b, r+1, t)$$
$$+ \rho r P(b+1, r, t) \tag{1}$$

For a state on the boundary where one of the sides has no survivors, only two types of transition are possible, and in the similar way, the corresponding differential equations are

$$\frac{dP(b, 0, t)}{dt} = \beta b P(b, 1, t) \tag{2}$$

$$\frac{dP(0,r,t)}{dt} = \rho r P(1,r,t) \tag{3}$$

Since the battle starts in the state (B_0, R_0), the following initial conditions are imposed.

$$P(B_0, R_0, 0) = 1$$

$$P(b, r, 0) = 0 \tag{4}$$

$$\text{for } (b, r) \neq (B_0, B_0)$$

also, reinforcement is not provided to either side, therefore

$$P(b, r, t) = 0 \tag{5}$$

$$\text{for } b > B_0 \text{ and } r > R_0$$

The system of equations (1), (2), (3) with the initial and boundary conditions (4) and (5) are sufficent for the solution of the Exponential Lanchester model.

Even the simplest Exponential Lanchester model presents difficulties with numerical solution becoming rapidly intractable[8].

A simulation technique has been developed which has given a fast implementation for the solution of an Exponential Lanchester model. The theoretical basis of the simulation approach is as follows.

Consider a state probability $P(b, r, t)$, then the probability of no casualties by time t, can be written down by the same way as in the formulation of equation (1) as :

$$\frac{d\mathcal{P}(b,r,t)}{dt} = -(\beta b + \rho r)\mathcal{P}(b,r,t)$$

which can be integrated to yield

$$\mathcal{P}(b,r,t) = e^{-(\beta b + \rho r)t}$$

Hence the probability that there is at least one casualty by time t is

$$1 - e^{-(\beta b + \rho r)t}$$

which is a negative exponential distribution, and the mean time to the first kill is given by $1/(\beta b + \rho r)$.

It is also easy to see that the conditional probability of the first casualty being on the Blue side is simply the ratio of the total Red kill rate to the total combined kill rate for Red and Blue.

The probability that the first battle casualty is to Blue is,

$$\frac{\rho r}{\beta b+\rho r}$$

In outline, the procedure of this simulation technique is firstly to sample from the negative exponential distribution to determine the time to the first kill and then to determine, by means of random number sampling, whether the casualty is Blue or Red and decrease the strength of the appropriate force. The resulting situation is then regarded as the start of a new battle (with one side having one fewer combatants) and the process is repeated continually until a stopping criterion is satisfied. Diagram 1 shows the process in the form of a flow diagram.

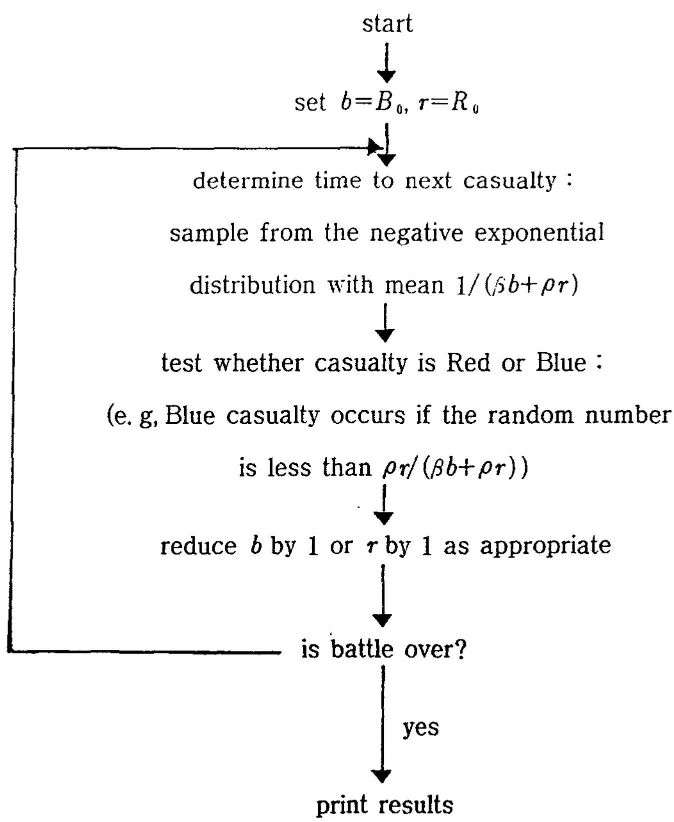


Diagram 1 : Flow Diagram of the Simulation Technique for the Solution of the Exponential Lanchester

3. 'BAGSIM' MODEL FOR THE INVESTIGATION OF THE LANCHESTER EQUATION

A Monte-Carlo combat simulation model has the potential to represent a combat situation in as much detail as the military analyst chooses. The Monte-Carlo simulation model for the battle group combat (called 'BAGSIM') has been developed by the author[3]. 'BAGSIM' is constructed to represent land combat at the battle group level, and allow many parameters and situations to be considered for the direct-fire land battle at this level. The 'BAGSIM' program has been written in Fortran to a structured design, and whilst complicated, it is clearly commented to allow the sequence of events to be followed easily. In particular, 'BAGSIM' is designed to be flexible and easily adapted to represent equivalent situations to the other more aggregated models. The analysis comparing 'BAGSIM' with some of the existing General Renewal models has shown that the 'BAGSIM' are working correctly[3], on the basis of which an evaluation of the applicability of the 'DL' and 'EL' models to the many-on-many direct-fire engagement is investigated.

4. APPLICABILITY OF THE DETERMINISTIC LANCHESTER MODEL TO THE MANY-ON-MANY ENGAGEMENT

4.1 Introduction

The Deterministic Lanchester model ('Square Law' model) is generally accepted as a reasonable representation of a many-on-many engagement in which the inter-firing times follow a negative exponential distribution, the initial number of combatants or the power ratio is large. However it would be rather dangerous to apply the 'DL' model to a general many-on-many engagement, particularly since there are no clear guidelines as to when the 'DL' model gives misleading results.

Therefore some clear guidelines for the application of the 'DL' model based on the following parameters are required.

- The initial number of combatants

- The power ratio
- The inter-firing time distribution

For this problem, the 'DL' model and 'BAGSIM' are compared under set of the equivalent situation consisting of initial number of combatants, single shot kill probability and inter-firing times. The corresponding data-sets are set up for the comparison at Annex I. A further explanation of the data-sets helps to understand easily the results of the comparison.

At first, 10 data-sets are considered and each data-set covers 19 cases of experiments, thus there are a total of 190 cases. The 10 data-sets are for the increasing size of the initial number of combatant. The first data-set, data-set 1, contains the initial number of combatants up to 10, data-set 2 contains initial numbers from 11 up to 20, data-set 3 contains initial numbers from 21 up to 30, etc.

The 19 cases in each data-set are for a different power ratio. Each case starts from the power ratio 1 to 10 increasing by steps of 0.5. The first case has the power ratio 1, the second case has 1.5, the third case has 2, etc. It is noted that data-set $i.j$ is referring to j^{th} case (numbered) at the data-set i.

The 'DL' model produces the following outputs : number of survivors at the end of the battle, combat time at the end of the battle and number of combatants at time t. In order to compare these outputs with those of the 'BAGSIM' model, a number of 'consistency' have been defined as follows.

1. The consistency for number of combatants at the end of the battle is defined as $C_n(X_n, Y_n) = 1 - |X_n - Y_n| / Y_n$, where X_n is number of survivors at the end of the battle from the model X (for example the 'DL' model), and Y_n is the same output from the model Y (for example the 'BAGSIM' model).
2. The consistency for combat time at the end of the battle is defined as $C_t(X_t, Y_t) = 1 - |X_t - Y_t| / Y_t$, where X_t is combat time at the end of the battle from the model X, and Y_t is the equivalent output from the model Y.

It is noted that when all the outputs to be compared do not have random property, using such a consistency value for the comparision could be more informative. For example if the consistency value is 1, then the models compared are certainly equal in terms of this output. Otherwise, using a statistics for the comparison of the outputs having random property could give rather reasonable comparative result.

4. 2 Comparison of the 'DL' Model and 'BAGSIM' Using the Negative Exponential Inter-Firing Time Distribution

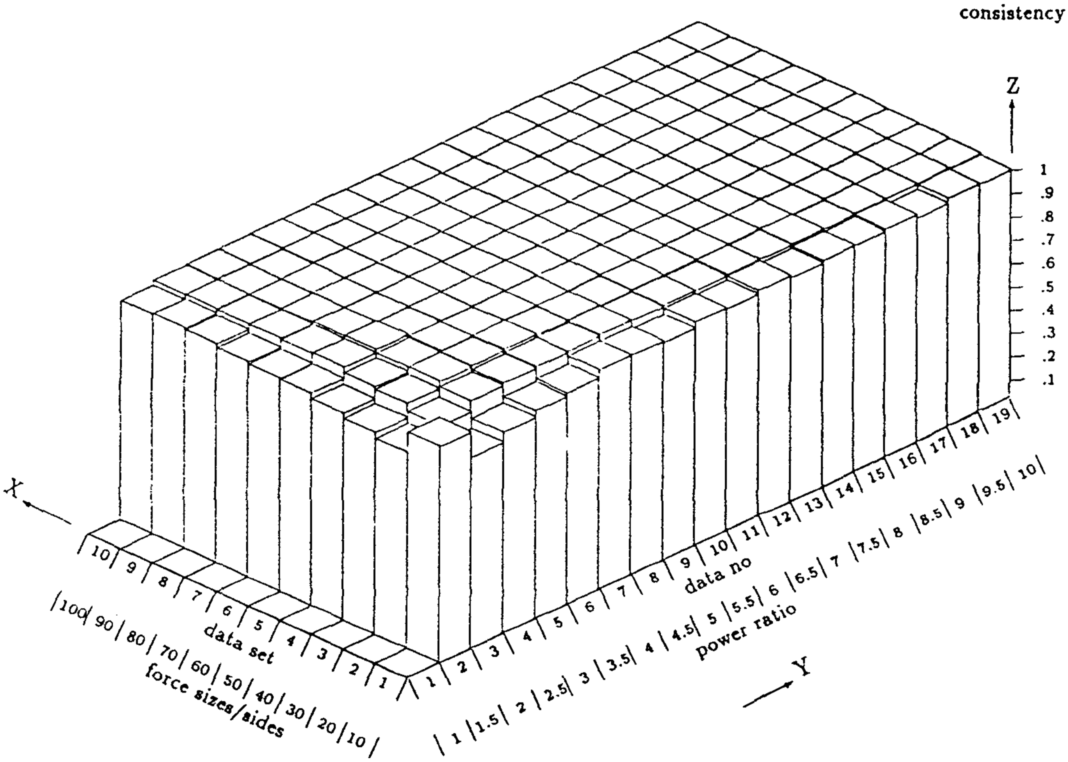
The 'DL' model has been compared with the 'BAGSIM' model using the ten data-sets shown at Annex I and discussed earlier. In 'BAGSIM' model, the inter-firing times have a negative exponential distribution, and its results are obtained from 5000 replications for an engagement.

The three-dimensional bar-chart, Figure 1, shows C_n ('DL', 'BAGSIM'), the consistency values between numbers of Blue survivors from the 'DL' model and those from the 'BAGSIM' model which was run with a negative exponential inter-firing time distribution. As shown in this bar-chart, the consistency values are less than 1 when the initial number of combatants and the power ratio are small, but it approaches 1 either as the initial number of combatants or the power ratio increases.

The three-dimensional bar-chart, Figure 2, shows C_t ('DL', 'BAGSIM'), the consistency valus between combat times from the 'DL' model and those from the 'BAGSIM' model. As shown in this bar-chart, the consistency values are also less than 1 when the initial number of combatants and the power ratio are small, but again it approaches 1 either as the initial number of combatants or the power ratio increases.

However, it is noted that even though the consistency value for numbers of survivors is near to 1, if the consistency value for combat times is much less than 1, then the two models compared are different. For example, in Figure 1 the consistency value for numbers of Blue survivors for data-set 1. 2 is 0. 9597, on the other hand the consistancy value for combat times is 0. 4200 as shown in Figure 2.

When numbers of Blue combatants at time t from the 'DL' model and the 'BAGSIM' model are plotted on a graph, it can be seen that they are different but that the relative differences become less as the initial number of combatants increases for a given power ratio. A typical example of these graphs is shown in Figures 3. Figure 3 is based on data-set 2.2 (number of combatants on each side is 12 and the power ratio is 1.5 in favour of Blue).

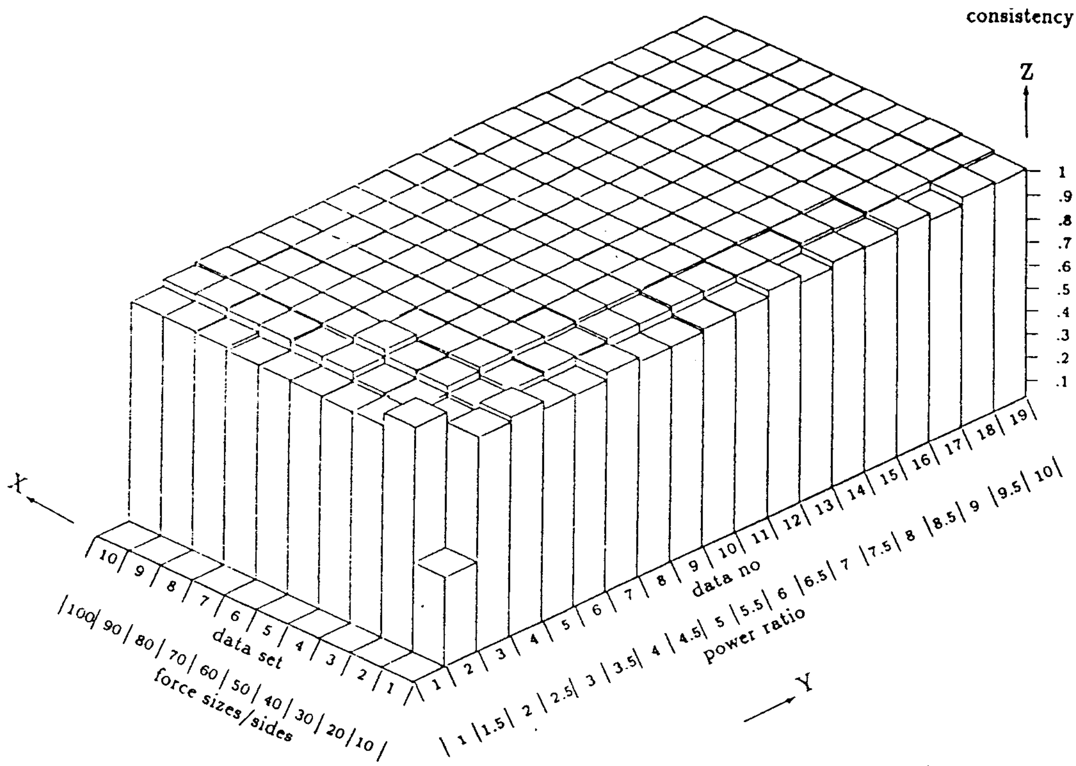


X : the data-sets (set 1~set 10) at Annex I

Y : the cases (case 1~case 19) given a data-set (the power ratio of which is from 1 to 10)

Z : consistency value for numbers of Blue survivors at the end of the battle between the 'DL' model and the 'BAGSIM' model run using the negative exponential inter-firing time (IFT) distribution.

Figure 1 : Comparison of the 'DL' Model and 'BAGSIM' Run Using the Negative Exponential IFT Distribution in Terms of Number of Blue Survivors at the End of the Battle



X : the data-sets (set 1~set 10) at Annex I

Y : the cases (case 1~case 19) given a data-set

(the power ratio of which is from 1 to 10)

Z : consistency value for combat times at the end of the battle between the 'DL' model and the 'BAGSIM' model run using the negative exponential IFT distribution .

Figure 2 : Comparison of the 'DL' Model and 'BAGSIM' Run Using the Negative Exponential IFT Distribution in Terms of Combat Time at the End of the Battle

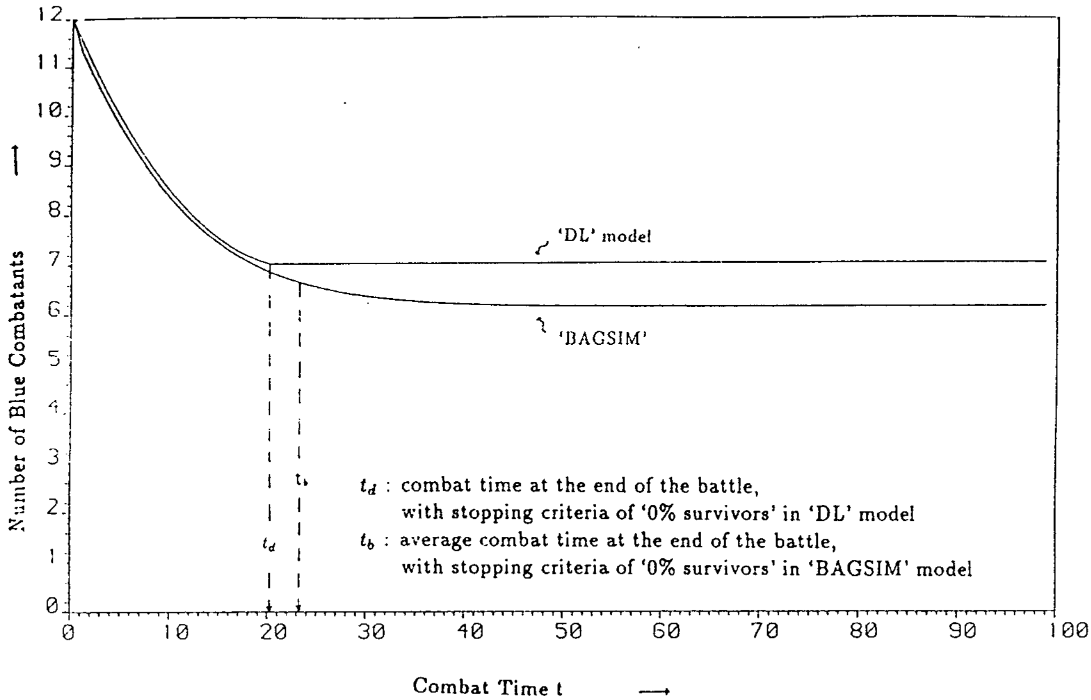
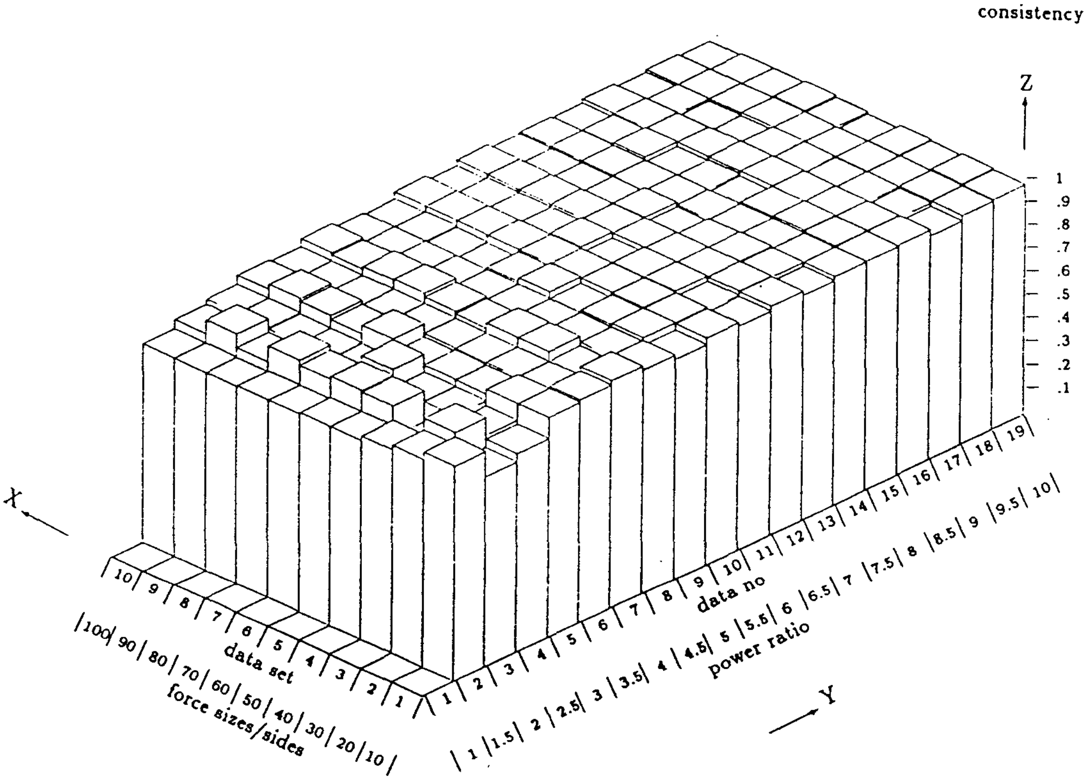


Figure 3 : Number of Blue Combatants at Time t in the 'DL' Model and 'BAGSIM' Run Using the Negative Exponential IFT Distribution (12 v 12)

4.3 Comparison of the 'DL' Model and 'BAGSIM' Using an Erlang Inter-Firing Time Distribution

The 'DL' model is again compared with the 'BAGSIM' model using the data-sets at Annex I, but this time using an Erlang distribution instead of a negative exponential distribution for the inter-firing times in 'BAGSIM'. The shape parameter of the Erlang distribution was taken to be 2, i. e. Erlang$(2, \theta)$, and the scale parameter θ is then obtained from the given mean value in the data-sets by putting $\theta=2/\text{mean}$.

The three-dimensional bar-chart, Figure 4, shows C_n('DL', 'BAGSIM'), the consistency values between numbers of Blue survivors from the 'DL' model and those from the 'BAGSIM' model which was run with an Erlang$(2, \theta)$ inter-firing time distribution. In this bar-chart, it is shown that the consistency values are less than 1 when the initial number of combatants and the power ratio are small, on the other hand, when the initial number of combatants and the power ratio become large many of the consistency values approach 1, but sometimes less than 0.9800 (for instance, data-sets 8.13, 9.13, 10.15).



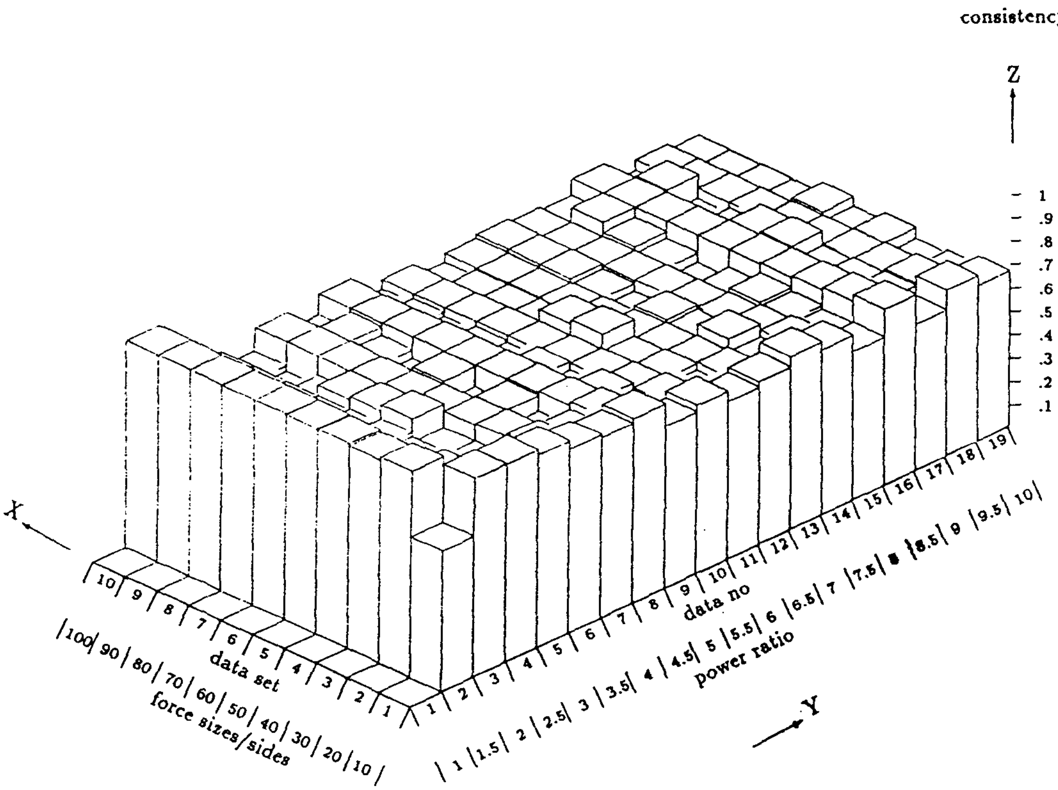
X : the data-sets (set 1-set 10) at Annex I

Y : the cases (case 1~case 19) given a data-set (the power ratio of which is from 1 to 10)

Z : consistency value of numbers of Blue survivors at the end of the battle between the 'DL' model and 'BAGSIM' run using the Erlang (2, θ) IFT distribution.

Figure 4 : Comparison of the 'DL' Model and 'BAGSIM' Run Using the Erlang (2, θ) IFT Distribution in Terms of Number of Blue Survivors at the End of the Battle

The three-dimensional bar-chart, Figure 5, shows C_t ('DL', 'BAGSIM'), the consistency values between combat times from the *'DL' model and those from the 'BAGSIM' model. In* this bar-chart, it is shown that most of the consistency values are less than 0.8000, even when the initial number of combatants and the power ratio increase.



X : the data-sets (set 1 ~ set 10) at Annex I

Y : the cases (case 1 ~ case 19) given a data-set

(the power ratio of which is from 1 to 10)

Z : consistency value of combat times at the end of the battle between the 'DL' model and the 'BAGSIM' model run using the Erlang $(2, \theta)$ IFT distribution.

Figure 5 : Comparison of the 'DL' Model and 'BAGSIM' Run Using the Erlang $(2, \theta)$ IFT Distribution in Terms of Combat Times at the End of the Battle

When numbers of Blue combatants at time t from the 'DL' model and the 'BAGSIM' model are plotted on a graph. It has shown that they are substantially different even when the initial number of combatants and the power ratio increase.

4. 4 Conclusion

The Deterministic Lanchester model has been compared with 'BAGSIM' for the 190 cases to establish guidelines for the application of the 'DL' model to the many-on-many engagement. This has been done under two situations : firstly when the inter-firing times follow the negative exponential distribution and secondly when they follow the Erlang distribution. From this analysis, it is concluded that :

1. The applicability of the 'DL' model to the many-on-many engagement based on the initial number of combatants and the power ratio is as shown in the shaded area of Figure 6. This is based on the inter-firing time being negative exponential and the resulting consistency value between the two models being at least 0. 9950.
2. On the other hand, the 'DL' model would give substantially different results when the inter-firing times have other than a negative exponential distribution.

5. Applicability of The Exponential Lanchester Model to The Many-On-Many Engagement

5. 1 Introduction

An Exponential Lanchester model is a stochastic formulation of a Deterministic Lanchester model, and the inter-firing times in the 'EL' model are implicitly assumed to have a negative exponential distribution.

Hence, it is reasonable to apply the 'EL' model to the many-on-many engagement where the inter-firing times follow a negative exponential distribution. Howerver, how good an approximation is it if the inter-firing times do not follow a negative exponential distribution?. To try and answer this question, the 'EL' model and 'BAGSIM' were compared under the same situation as in the previous section, using once again the data-sets at Annex I.

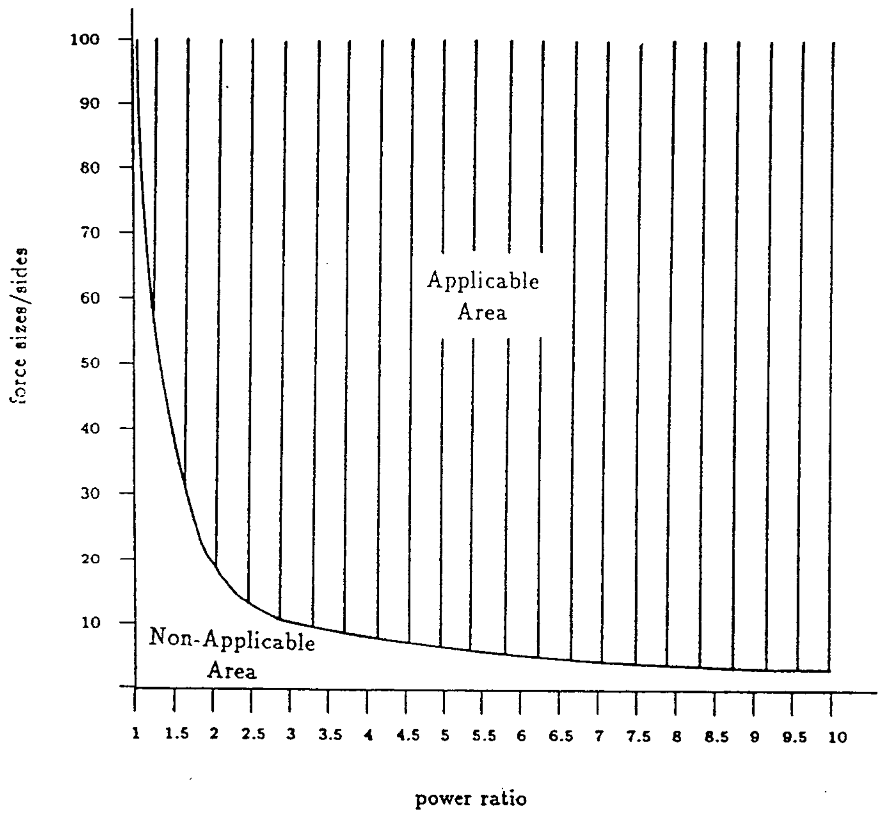


Figure 6 : Applicability of the 'DL' Model to the Many-on-Many Engagement Based on the Initial Number of Combatants of Both Sides and the Power Ratio.

5.2 Comparison of the Exponential Lanchester 'EL' Model and 'BAGSIM' Using the Negative Exponential Inter-Firing Time Distribution

The 'EL' model has been compared with 'BAGSIM' using the ten data-sets shown at Annex I. For the solution of the 'EL' model, the simulation technique has been applied. In the 'BAGSIM' model, the inter-firing times have a negative exponential distribution.

The Table 1 shows win probabilities from the 'EL' model and those from the 'BAGSIM' model which was run with a negative exponential inter-firing time distribution, and the relevant

Results

Case	'EL' Model (Simulated Results)		'BAGSIM' (Simulated Results)		d(†)	p(‡)
	Pr(Blue win)	Pr(Red win)	Pr(Blue win)	Pr(Red win)		
1	.5040	.4960	.5040	.4960	0	1.00
2	.6546	.3454	.7104	.2896	6.00	0.00
3	.7802	.2198	.8102	.1898	3.72	0.00
4	.8426	.1574	.8598	.1402	2.42	0.02
5	.9274	.0726	.9530	.0470	5.40	0.00
6	.9426	.0574	.9832	.0168	10.74	0.00
7	.9680	.0320	.9888	.0112	7.15	0.00
8	.9776	.0224	.9966	.0034	8.42	0.00
9	.9740	.0260	.9884	.0116	5.30	0.00
10	.9950	.0050	.9984	.0016	2.96	0.00

† d=the standardised quantity given by

$$\left| \frac{P_1 - P_2}{\sqrt{\bar{P}(1-\bar{P})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \right|$$

where P_1 : Pr (Blue win) in 'EL' Model

P_2 : Pr (Blue win) in 'BAGSIM'

n_2, n_2 : number of replications for an engagement in the 'EL' and 'BAGSIM' respectively

$$\bar{P} = \frac{P_1 + P_2}{2}$$

‡ p=the probability of obtaining a value as large as or larger than the standardised quantity +d

Table 1 : Comparison of the 'EL' Model and 'BAGSIM' Run Using the Negative Exponential IFT Distribution

statistics for 'Significance Test for Two Unknown Proportions' for some 10 cases using the data-set 1.1 to 1.10. As can be seen from this table, they are equivalent to each other.

All of the consistency values not only between number of Blue survivors but also between combat times, from the 'EL' model and the 'BAGSIM' model, have shown to be larger than 0.9950. Further, when number of combatants at time t from the two models are plotted on a graph, they have shown to be very closely fitted.

The simulated results are based on 5000 replications for an engagement.

The p values in the above cases are all larger than 0.05 (i.e. 5% significance level), hence the null hypothesis that the win probabilities of the 'EL' model and 'BAGSIM' are equal is accepted at the 5% level.

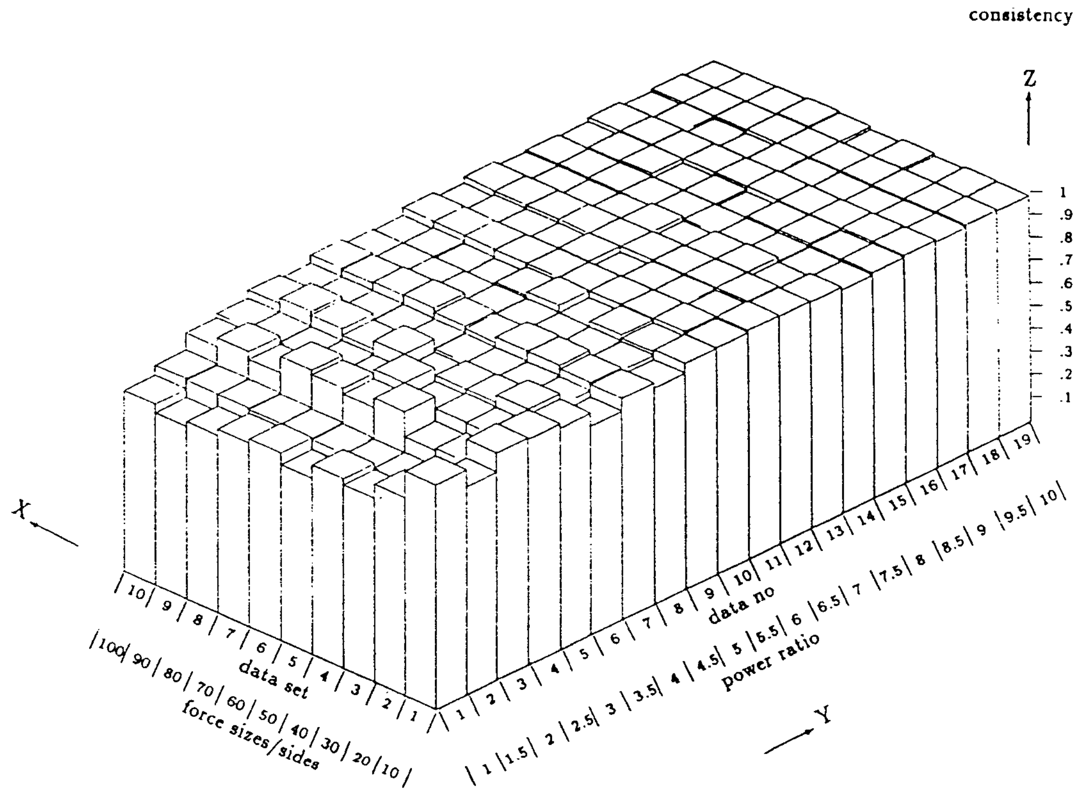
5.3 Comparison of the Exponential Lanchester Model and 'BAGSIM' Using an Erlang Inter-Firing-Time Distribution

The 'EL' model is again compared with the 'BAGSIM' model using the data-sets at Annex I, but this time using an Erlang$(2, \theta)$ distribution instead of a negative exponential distribution for the inter-firing times in 'BAGSIM'

The 'Significance Test for Two Unknown Proportions' has been done for the same 10 cases as in Table 1. From this most of win probabilities has been significantly different from each other. Furthermore most of the consistency values for the number of Blue survivors and the combat times have shown to be much less than 1.

The three-dimensional bar-chart, Figure 7, shows C_n('EL', 'BAGSIM'), the consistency values between numbers of Blue survivors from the 'EL' model and those from the 'BAGSIM' model, for the 190 cases using whole data-sets at Annex I. In this bar-chart, it is shown that the consistency values are less than 1. However when the initial number of combatants and the power ratio increase, many of the consistency values approach 1, but less than 0.9800.

The three-dimensional bar-chart, Figure 8, shows C_t('EL', 'BAGSIM'), the consistency values between combat times from the 'EL' model and those from the 'BAGSIM' model for the 190 cases using whole data-sets at Annex I. In this bar-chart, it is shown that most of the consistency values are much less than 1.

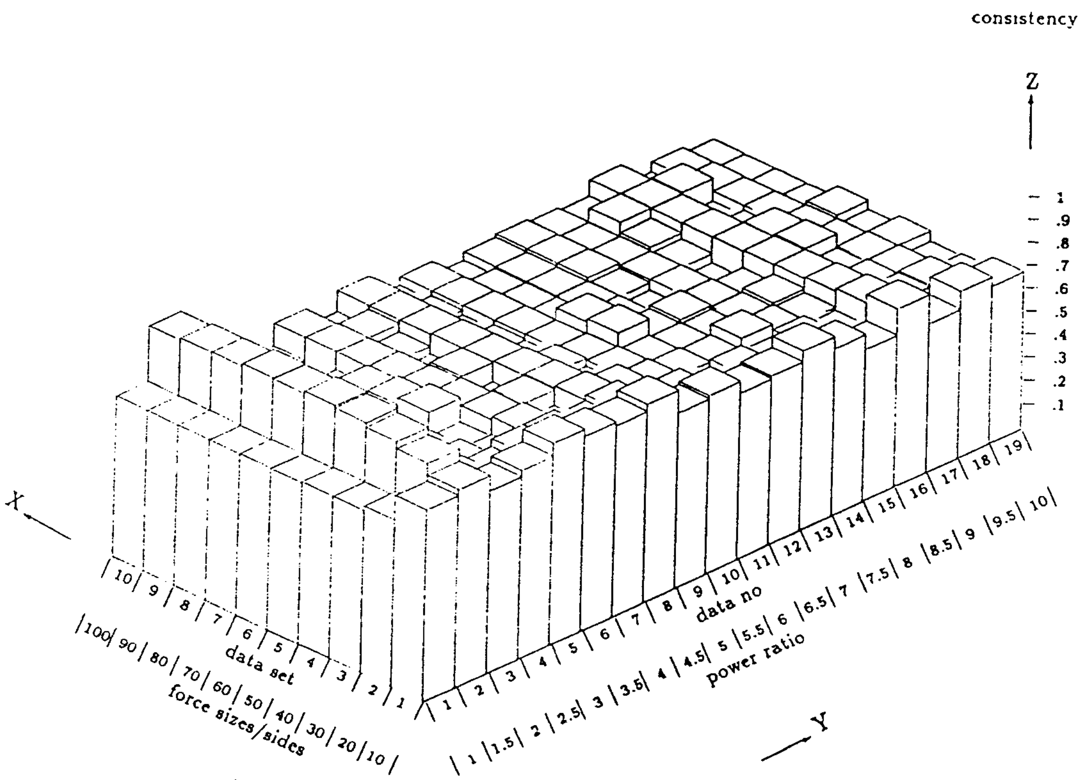


X : the data-sets (set 1 ~ set 10) at Annex I

Y : the cases (case 1 ~ case 19) given a data-set
(the power ratio of which is from 1 to 10)

Z : consistency value for numbers of Blue survivors at the end of the battle between the 'EL' model and the 'BAGSIM' model run using the Erlang (2, θ) IFT distribution

Figure 7 : Comparison of the 'EL' Model and 'BAGSIM' Run Using the Erlang (2, θ) IFT Distribution in Terms of Number of Blue Survivors at the End of the Battle



X : the data-sets (set 1 ~ set 10) at Annex I

Y : the cases (case 1 ~ case 19) given a data-set

(the power ratio of which is from 1 to 10)

Z : consistency value for combat times at the end of the battle between the 'EL' model and the 'BAGSIM' model run using the Erlang(2, θ) IFT distribution

Figure 8 : Comparison of the 'EL' Model and 'BAGSIM' Run Using the Erlang(2, θ) IFT Distribution in Terms of Combat Time at the End of the Battle

When numbers of Blue combatants at time t from the 'EL' model and the 'BAGSIM' model are plotted on a graph, they have poorly matched regardless of the size of the initial number of combatants and the power ratio.

5.4 Conclusions

From the above analysis on the Exponential Lanchester model, it is concluded as follows.

The 'EL' model has given exactly the same results as those of the 'BAGSIM' model which was run with the negative exponential inter-firing time distribution. However the 'EL' model and the 'BAGSIM' model with the Erlang$(2, \theta)$ inter-firing time distribution have produced considerably different results. Hence, the 'EL' model is not a good approximation to a general many-on-many engagement.

6. CONCLUSION

In this paper, the applicability of the traditional Deterministic and Exponential Lanchester models to the many-on-many direct-fire engagement on land has been examined. From the work it is concluded that the Deterministic Lanchester model is reasonable for the many-on-many direct-fire engagement when the initial numbers of both sides are more than 20, the power ratio is more than 2, and the inter-firing times follow a negative exponential distribution. If the inter-firing time distributions are other than negative exponential, the Deterministic Lanchester model may not be reasonable even for large initial numbers and power ratio. The Exponential Lanchester model represents the special case when the inter-firing times follow a negative exponential distribution. Further it is not a good approximation to the general many-on-many engagement.

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Annex I : Data-Sets for the Investigation of the Applicability of the 'DL'and 'EL'Models to the Many-on-Many Engagement

This Annex contains the 10 data-sets. Each data-set covers 19 cases for the many-on-many engagement, all of which have a different power ratio.

Notation :

No. - sequence of values in each data-set

B_0 - initial number of the Blue side

R_0 - initial number of the Red side

p - single shot kill probability of each Blue

$\tilde{p}$ - single shot kill probability of each Red

a - mean inter-firing time of each Blue

$\tilde{a}$ - mean inter-firing time of each Blue

w - power ratio ($\beta B_0^2/\rho R_0^2$)

where,

$\beta = p/a$

$\rho = \tilde{p} / \tilde{a}$

Note that data-set i. j is referring to jth case at the data-set i.

	Data-Set 1							Data-Set 2						
No.	B_0	R_0	p	$\tilde{p}$	a	$\tilde{a}$	ω	B_0	R_0	p	$\tilde{p}$	a	$\tilde{a}$	ω
1	1	1	.7	.7	7	7	1	11	11	.7	.7	7	7	1
2	2	2	.3	.6	5	15	1.5	12	12	.3	.6	5	15	1.5
3	4	3	.5	.5	8	9	2	11	11	.9	.9	5	10	2
4	3	3	.7	.2	7	5	2.5	13	13	.7	.2	7	5	2.5
5	4	5	.5	.2	8	15	3	12	14	.7	.2	6	7	3
6	4	4	.7	.6	5	15	3.5	14	14	.7	.6	5	15	3.5
7	4	6	.9	.2	5	10	4	12	14	.7	.1	9	7	4
8	5	5	.6	.4	5	15	4.5	15	15	.6	.4	5	15	4.5
9	5	3	.7	.5	7	9	5	15	12	.8	.5	6	12	5
10	6	6	.8	.2	8	11	5.5	16	16	.8	.2	8	11	5.5
11	6	4	.8	.3	8	8	6	15	10	.8	.3	6	6	6
12	7	7	.9	.2	9	13	6.5	17	17	.9	.2	9	13	6.5
13	7	5	.5	.1	7	5	7	15	10	.8	.6	6	4	7
14	8	8	.9	.3	6	15	7.5	18	18	.9	.3	6	15	7.5
15	8	4	.8	.6	6	9	8	16	12	.9	.4	5	10	8
16	9	9	.6	.1	12	17	8.5	19	19	.6	.1	12	17	8.5
17	9	3	.6	.5	6	5	9	18	14	.7	.1	9	7	9
18	10	10	.8	.1	16	19	9.5	20	20	.8	.1	16	19	9.5
19	9	6	.8	.3	6	10	10	21	12	.8	.3	6	10	10

	Data-Set 3							Data-Set 4						
No.	B_0	R_0	p	$\tilde{p}$	a	$\tilde{a}$	ω	B_0	R_0	p	$\tilde{p}$	a	$\tilde{a}$	ω
1	21	21	.7	.7	7	7	1	31	31	.7	.7	7	7	1
2	22	22	.3	.6	5	15	1.5	32	32	.3	.6	5	15	1.5
3	24	20	.5	.2	9	5	2	35	30	.8	.7	7	9	2
4	23	23	.7	.2	7	5	2.5	33	33	.7	.2	7	5	2.5
5	27	24	.8	.3	9	8	3	36	32	.8	.3	9	8	3
6	24	24	.7	.6	5	15	3.5	34	34	.7	.6	5	15	3.5
7	25	20	.8	.5	5	8	4	36	30	.5	.3	9	15	4
8	25	25	.6	.4	5	15	4.5	35	35	.6	.4	5	15	4.5
9	28	21	.5	.2	8	9	5	35	30	.9	.7	7	20	5
10	26	26	.8	.2	8	11	5.5	36	36	.8	.2	8	11	5.5
11	27	24	.8	.3	9	16	6	36	32	.8	.3	9	16	6
12	27	27	.9	.2	9	13	6.5	37	37	.9	.2	9	13	6.5
13	28	20	.5	.2	7	10	7	35	30	.8	.2	7	8	7
14	28	28	.9	.3	6	15	7.5	38	38	.9	.3	6	15	7.5
15	28	21	.7	.2	8	15	8	35	35	.8	.2	5	10	8
16	29	29	.6	.1	12	17	8.5	39	39	.6	.1	12	17	8.5
17	27	21	.7	.3	6	14	9	36	32	.8	.3	6	16	9
18	30	30	.8	.1	16	19	9.5	40	40	.8	.1	16	19	9.5
19	28	21	.9	.4	6	15	10	37	37	.8	.1	8	10	10

	Data-Set 5							Data-Set 6						
No.	B_0	R_0	p	$\tilde{p}$	a	$\tilde{a}$	ω	B_0	R_0	p	$\tilde{p}$	a	$\tilde{a}$	ω
1	41	41	.7	.7	7	7	1	51	51	.7	.7	7	7	1
2	42	42	.3	.6	5	15	1.5	52	52	.3	.6	5	15	1.5
3	45	40	.8	.9	9	16	2	60	60	.5	.4	9	10	2
4	43	43	.7	.2	7	15	2.5	53	53	.7	.2	7	5	2.5
5	48	40	.5	.8	6	20	3	60	50	.5	.4	6	10	3
6	44	44	.7	.6	5	15	3.5	54	54	.7	.6	5	15	3.5
7	49	42	.9	.7	7	16	4	60	50	.5	.4	9	20	4
8	45	45	.6	.4	5	15	4.5	55	55	.6	.4	5	15	4.5
9	50	40	.6	.3	5	8	5	60	54	.9	.5	8	18	5
10	46	46	.8	.2	8	11	5.5	56	56	.8	.2	8	11	5.5
11	48	40	.5	.4	6	20	6	60	50	.5	.2	6	10	6
12	47	47	.9	.2	9	13	6.5	57	57	.9	.2	9	13	6.5
13	50	40	.7	.5	5	16	7	60	45	.9	.4	8	14	7
14	48	48	.9	.3	6	15	7.5	58	58	.9	.9	6	15	7.5
15	50	45	.9	.5	5	18	8	60	45	.9	.2	8	8	8
16	49	49	.6	.1	12	17	8.5	59	59	.6	.1	12	17	8.5
17	48	40	.5	.2	8	20	9	60	50	.5	.1	8	10	9
18	50	50	.8	.1	16	19	9.5	60	60	.8	.1	16	19	9.5
19	50	45	.9	.4	5	18	10	60	48	.6	.1	15	16	10

	Data-Set 7							Data-Set 8						
No.	B_0	R_0	p	$\tilde{p}$	a	$\tilde{a}$	ω	B_0	R_0	p	$\tilde{p}$	a	$\tilde{a}$	ω
1	61	61	.7	.7	7	7	1	71	71	.7	.7	7	7	1
2	62	62	.3	.6	5	15	1.5	72	72	.3	.6	5	15	1.5
3	70	60	.8	.7	7	9	2	80	70	.7	.4	8	7	2
4	63	63	.7	.2	7	5	2.5	73	73	.7	.2	7	5	2.5
5	70	60	.6	.7	7	18	3	80	64	.8	.5	5	6	3
6	64	64	.7	.6	5	15	3.5	74	74	.7	.6	5	15	3.5
7	70	56	.8	.5	5	8	4	80	70	.7	.4	8	14	4
8	65	65	.6	.4	5	15	4.5	75	75	.6	.4	5	15	4.5
9	70	63	.9	.8	5	18	5	80	64	.9	.9	5	16	5
10	66	66	.8	.2	8	11	5.5	76	76	.8	.2	8	11	5.5
11	70	56	.8	.5	5	12	6	80	64	.8	.5	5	12	6
12	67	67	.9	.2	9	13	6.5	77	77	.9	.2	9	13	6.5
13	70	56	.8	.5	5	14	7	80	64	.7	.5	5	16	7
14	68	68	.9	.3	6	14	7.5	78	78	.9	.3	6	15	7.5
15	70	56	.8	.5	5	16	8	80	70	.7	.2	8	14	8
16	69	69	.6	.1	12	17	8.5	79	79	.6	.1	12	17	8.5
17	70	56	.8	.5	6	18	9	80	64	.9	.5	5	16	9
18	70	70	.8	.1	16	19	9.5	80	80	.8	.1	16	19	9.5
19	70	56	.8	.2	6	8	10	80	64	.8	.3	5	12	10

	Data-Set 9							Data-Set 10						
No.	B_0	R_0	p	$\tilde{p}$	a	$\tilde{a}$	ω	B_0	R_0	p	$\tilde{p}$	a	$\tilde{a}$	ω
1	81	81	.7	.7	7	7	1	91	91	.7	.7	7	7	1
2	82	82	.3	.6	5	15	1.5	92	92	.3	.6	5	15	1.5
3	90	80	.8	.9	9	16	2	100	90	.9	.5	10	9	2
4	83	83	.7	.2	7	5	2.5	93	93	.7	.2	7	5	2.5
5	90	60	.9	.9	9	12	3	100	80	.9	.5	15	16	3
6	84	84	.7	.6	5	15	3.5	94	94	.7	.6	5	15	3.5
7	90	75	.5	.2	9	10	4	100	90	.9	.5	10	18	4
8	85	85	.6	.4	5	15	4.5	95	95	.6	.4	5	15	4.5
9	90	72	.8	.7	5	14	5	100	80	.9	.9	5	16	5
10	86	86	.8	.2	8	11	5.5	96	96	.8	.2	8	11	5.5
11	90	72	.8	.5	5	12	6	100	75	.9	.8	5	15	6
12	87	87	.9	.2	9	13	6.5	97	97	.9	.2	9	13	6.5
13	90	72	.7	.5	5	16	7	100	80	.7	.5	5	16	7
14	88	88	.9	.3	6	15	7.5	98	98	.9	.3	6	15	7.5
15	90	75	.5	.1	9	10	8	100	80	.8	.5	5	16	8
16	89	89	.6	.1	12	17	8.5	99	99	.6	.1	12	17	8.5
17	90	75	.5	.2	6	15	9	100	80	.9	.5	5	16	9
18	90	90	.8	.1	16	19	9.5	100	100	.8	.1	16	19	9.5
19	90	72	.8	.3	5	12	10	100	80	.8	.4	5	16	10