Journal of the Korean Institute
of Industrial Engineers
Vol. 15, No. 2, Dec., 1989

Analysis of Non-linear Quantity Discount for Heterogeneous Characteristics.

Kyung Keun Lee*

상이한 복수고객에 대한 비선형 가격할인

이 경 근*

Abstract

From the supplier's point of view, we examine the existence of a Pareto superior pricing schedule for one wholesaler with multiple retailers. In the case of multiple retailers, an order quantity pricing schedule should depends on the retailer's underlying characteristics. But identification of each retailer's characteristics may be impossible; rather, the wholesaler knows only the probability distribution of each ratailer's characteristics. Perfect price discrimination is impossible because a separate pricing schedule cannot be tailored for each retailer. Some degree of discrimination is possible only by using a non-linear pricing schedule. From this analysis based on the non-linear pricing, we conclude that there is no Pareto superior pricing schedule for the case of multiple retailers.

1. Introduction

Several papers have been written on the subjects of inventory control involving lot sizing with quantity discount. This is an important subject, given the wide-spread use of quantity discounts in industry nowdays. The traditional quantity discount models, however, have usually been studied soley from the point of view of the retailer, not from the view of the supplier.

Monahan[1984], Rosenblatt and Lee [1985, 1986], Eppen and Lieberman[1984] have studied the important implications

* Department of Industrial Engineering, Pusan National University

from the supplier's point of view of offering a quantity discount to his sole major retailers. Monahan shows that, by offering an all-unit discount schedule with a single break, the supplier can always improve his profit. This parallels the finding of Garbor [1955] on the sufficiency of single break quantity discount pricing schedule for the perfect price discrimination motivation with homogeneous customers. Lee and Rosenblatt have generalized the result of Monahan's model to add constraints imposed on the amount of discount and to drop the lot-for-lot assumption of the supplier's order replenishment. Eppen and Liebermann assume that retailers with the same demands can be separated into two groups. The first group consists of retailers who have low inventory holding cost. In the second group are retailers who have high inventory holding cost. They show that quantity discount are important if a high percentage of retailers will buy on deal for a small discount.

Kim and Hwang[1988] developed a single break incremental pricing schedule for the multiple retailers case with the assumption on the supplier's perfect information on the retailer's characteristics.

But all the above studies implicitly assume that the supplier has perfect information on the retailer's characteristics.

Goyal[1976] and Lal and Staelin[1984] have taken different approach through an integrated inventory model for a marketing cannnel of distribution where the total inventory related costs are jointly minimized.

We would like to extend the above studies to one wholesaler with multiple retailres classified by their characteristics such as unit inventory holding cost, demand rate or ordering cost. In the case of multiple retailers with heterogeneous characteristics, the quantity dependent pricing schedule should, if possible, depend on each retailer's underlying characteristics. But identification of each retailer's characteristics may be impossible; rather, it is more reasonable to assume that the wholesaler knows the probability distribution function of retailer's characteristics. Perfect price discrimination is impossible because a separate pricing schedule cannot be tailored to different retailers. Nonetheless some degree of price discrimination is possible by using a non-linear quantity dependent pricing schedule. Any sign of disparities among retailer's order, however, may indicate an opportunity for beneficial price discrimination. Oren, Smith and Wilson [1982, 1983] and Goldman, Leland and Sibley[1984] achieved powerful insight into the qualitative properties of the optimal non-linear pricing schedules in the case of a monopoly supplier. Oren, Smith and Wilson [1983] also present results for a competitive situation.

In this paper, we apply the theory of non-linear pricing to clarify the existence of a Pareto superior quantity dependent pricing for a wholesaler with heterogeneous characteristics in multiple retailers. This non-linear pricing schedule induces price discrimination based on the "self-selection" of optimal ordering quantities by various retailer types, that is, the retailer always responds to this pricing schedule by optimizing his objective function. Thus, the implementation of such a schedule does not require that the wholesaler have perfect information about the retailer's characteristics.

2. Model Formulation

Let's consider the situation in which many retailers receive their replenishment stock from one wholesaler. The wholesaler, in return, receives his supply through order from the manufacturer or the other vendor.

Assuming that the retailers always respond to the non-linear pricing schedule by the self-selection of optimal order quantity for that schedule, it may be the wholesaler's advantage to offer a quantity discount pricing schedule even though the demand rate is constant because the economic order quantity of the retailers may change in order to minimize their total cost, consisting of purchasing cost, ordering cost and inventory holding cost. This may affect the total expected cost of the wholesaler and result in a higher expected profit.

In this situation, we assume that the ordering cost and demand rate of the retailers are the same and the wholesaler knows the probability distribution function of the unit inventory holding cost for the retailers. Under some conditions, a non-linear pricing schedule can benefit both the retailers and the wholesaler by transferring part of the inventory holding cost from the wholesaler to the retailers in return for the quantity discount.

As a result, both parties can benefit economically from this use of a nonlinear pricing schedule, although under some conditions, no mutually beneficial pricing schedule exists.

The following assumptions will be used for our analysis.

(1) Demand rate is constant and insensitive to price changes.

(2) No shortages are allowed on either side.

(3) Instantaneous replenishment at both the wholesaler and the retailer.

(4) The wholesaler knows the probability distribution of the retailer's unit inventory holding cost.

(5) Demand rate and ordering cost are constant and are same for all the retailers.

(6) A continum of many infinitesimally small retailers.

The following notations will be used.

$P(q)$ = Total purchasing cost for q units for the retailer from the wholesaler.

$C(q)$ = Total purchasing cost for q units for the wholesaler from the manufacturer or the other vendor.

D = Annual demand rate for each retailer.

K_s = Wholesaler's ordering cost per order.

K_r = Retailer's ordering cost per order.

h_s = Wholesaler's inventory holding cost per unit per year.

h_r=Retailer's inventory holding cost per unit per year.

$f(h_r)$=Probability distribution function of retailer's unit inventory holding cost.

p=Present fixed unit price offered to the retailers by the wholesaler.

2.1. Retailer's Behavior

If there exists only one wholesaler, the retailers always respond to the price schedule offered by the wholesaler. The retailer whose unit inventory holding cost is h_r has an optimal order quantity $q^*(h_r)$ that is obtained by minimizing his average total annual cost, i.e, total of purchasing cost, ordering cost and average inventory holding cost.

$$q^*(h_r)=\arg\ \mathrm{Min}\left\{p(q(h_r))\ \frac{D}{q(h_r)} + K_r\ \frac{D}{q(h_r)} + \frac{q(h_r)}{2}\ h_r\right\} \cdots\cdots (2.1)$$

Thus, the first necessary condition for Eq. (2.1), that is, retailer's self-selection condition, can be written as

$$\frac{d}{dq(h_r)}\{p(q(h_r))\}\frac{D}{q(h_r)} - \{p(q(h_r)) + K_r\}\ \frac{D}{q(h_r)^2} + \frac{h_r}{2} = 0 \cdots\cdots (2.2)$$

2.2. Wholesaler's Behavior

The wholesaler's goal is to maximize his expected net profit subject to the retailer's optimal behavior described above. Given the supply cost function C(q), the wholesaler is to select an optimal non-linear pricing schedule P(q) for his goal. We will assume that P(q) is positive and differential for the possible range of order quantity corresponding to the minimum and maximum unit inventory holding cost in the probability distribution function.

The decrease in the wholesaler's inventory level from his economic order quantity(q^*_s) is composed of the n(the number of the retailers) different ordering quantities of the retailers. In order to meet the demand from the retailer without shortage, the wholesaler must replenish his economic order quantity earlier than his economic ordering cycle just at the time when the order from a retailer is greater than his remaining inventory. Then the average inventory level for the wholesaler can be approximately expressed as $\frac{q^*_s}{2}$(assuming n is large). Hence, the expected annual net profit of the wholesaler is given by the expected value of his gross revenue minus his purchasing cost and ordering cost and inventory holding cost.

$$\mathrm{Max}\ \pi = n\int_{h_0}^{h_1} p(q(h_r))\ \frac{D}{q(h_r)}\ f(h_r)\,dh_r - C(q^*_s)\ \frac{nD}{q^*_s} - K_s\ \frac{nD}{q^*_s} - \frac{q^*_s}{2}h_s \cdots\cdots (2.3)$$

$$\left[\begin{array}{l} h_1=\text{maximum } h_r \text{ in } f(h_r) \\ h_0=\text{minimum } h_r \text{ in } f(h_r) \end{array}\right]$$

From the above, we know that the expected gross revenue is independent from the wholesaler's economic order quantity. Then wholesaler's economic order quantity can be independently determined just by minimizing his total cost.

Finally, the wholesaler's problem is to find non-linear pricing schedule P(q) to maximize his expected net profit subject to the optimal retailer's behavior, i.e,

$$\text{Max } \pi = n \int_{h_0}^{h_1} p(q(h_r)) \frac{D}{q(h_r)} f(h_r)dh_r - \Omega \quad \cdots\cdots (2.4)$$

$$\text{s.t. } \frac{d}{dq(h_r)}\{p(q(h_r))\}\frac{D}{q(h_r)} - \{p(q(h_r)) + K_r\} \frac{D}{q(h_r)^2} + \frac{h_r}{2} = 0$$

$$\Omega = C(q^*_s)\frac{nD}{q^*_s} - K_s \frac{nD}{q^*_s} - \frac{q^*_s}{2}h_s$$

$$= \text{constant value}$$

3. Analysis of Model

We now turn to the analysis of the model to obtain a pricing policy P(q), maximizing the wholesaler's expected gross revenus, and investigate whether this pricing schedule is beneficial to both parties or if it is possible to get such a pricing schedule that would benefit all parties. We can rewrite the wholesaler's problem to maximize the first part of Eq(2.4) without the constant n as follow.

$$\text{Max } \pi' = \int_{h_0}^{h_1} p(q(h_r)) \frac{D}{q(h_r)} f(h_r)dh_r \quad \cdots\cdots (3.1)$$

Integrating Eq.(3.1) by parts and substituting the retailer's self-selection condition given by Eq.(2.2) yields

$$\pi' = p(q(h_r)) \frac{D}{q(h_r)} F(h_r) \Big|_{h_0}^{h_1} + \int_{q_0}^{q_1} F(h_r) \left\{\frac{d}{dq}(p(q)) \frac{D}{q} - p(q) \frac{D}{q^2}\right\} dq$$

$$= p(q_0) \frac{D}{q_0} + \int_{q_0}^{q_1} F(h_r) \left\{\frac{K_r D}{q^2} - \frac{h_r}{2}\right\} dq \quad \cdots\cdots (3.2)$$

$$\begin{bmatrix} q_0 = \text{minimum } q(h_r) = q(h_1) \\ q_1 = \text{maximum } q(h_r) = q(h_0) \end{bmatrix}$$

For $q_0 < q < q_1$, Eq.(3.2) can be maximized with respect to h_r by pointwise maximization of the integrand. Thus, the Euler's first order necessary condition for an interior local maximum is

$$f(h_r)\left\{\frac{K_r D}{q^2} - \frac{h_r}{2}\right\} - \frac{1}{2} F(h_r) = 0 \quad \cdots\cdots (3.3)$$

From the above equation, we can find the relation of the retailer's order quantity and the unit inventory holding cost to determine the pricing schedule. By substituting this relation $h_r = h(q)$ from Fq.(3.3), i.e., inverse mapping of the optimum order quantity for each unit inventory holding cost, into the retailer's self-selection condition Eq.(2.2), we will get

$$\frac{d}{dq} p(q) = \frac{p(q)}{q} + \frac{K_r}{q} - \frac{q}{2D} h(q) \quad \cdots\cdots (3.4)$$

Boundary conditions involving q_0 and q_1 are obtained by examining Eq.(3.2). Since Eq.(3.2) is expressed parametrically on q_0 and q_1, necessary conditions for maximizing or minimizing with respect to q_0 and q_1 are

$$\frac{d}{dp_0}\{p(q_0)\}\frac{D}{q_0} - p(q_0)\frac{D}{q_0^2} - F(h_1)\left\{\frac{K_r D}{q_0^2}\right\} = 0 \quad \cdots\cdots (3.5)$$

$$F(h_0)\left\{\frac{K_rD}{q_1^2}-\frac{h_0}{2}\right\}=0 \quad \cdots\cdots\cdots\cdots\cdots (3.6)$$

(where $F(h_0)=0$ and $F(h_1)=1$)

From Eqs(3.5) and (3.6),

$$\frac{d}{dq_0}\,p(q_0)=\frac{p(q_0)}{q_0}+\frac{K_r}{q_0}-\frac{h_1}{2}\,\frac{q_0}{D} \quad \cdots (3.7)$$

Therefore the boundary condition involving q_0 is exactly the same as the retailer's self-selection condition Eq.(2.2) while the boundary condition involving q_1 is always true. Solving the linear differential equation Eq.(3.4), we will get the P(q) as follow.

$$p(q)=-K_r-\frac{q}{2D}\int h(q)\,dq+C_0q \quad \cdots (3.8)$$

(where C_0: arbitrary constant number from integration.)

Because we assumed that total demand is fixed, a profit maximization criterion will drive up C_0 to infinity, which is clearly unreasonable. We must, therefore, recognize that the fixed demand assumption is reasonable only as long as prices stay within an acceptable range and not attempt to use profit maximization in determining C_0. Instead, we will use a satisfying approach and attempt to impose some restrictions on C_0. In particular, we would like to determine C_0 such that

(1) Wholesaler's expected net profit with the new pricing policy is greater than his expected net profit with existing fixed unit price schedule(old pricing policy).

(2) Retailer's total cost with the new pricing policy is less than his total cost with the existing fixed unit price schedule.

If it is possible to find such a policy, this will produce a Pareto improvement over the fixed unit pricing policy, which will benefit both the wholesaler and the retailers.

3.1. Wholesaler's Expected Profit

If the wholesaler's expected net profit with the new pricing policy(π_{new}) is less than his expected net profit with the old pricing policy(π_{old}), this new pricing policy is unacceptable, and he will not offer it. So this wholesaler's expected increase in profit is a necessarey condition for the problem. First, the wholeslaer's expected net profit with the old pricing policy can by written as follow.

$$\pi_{old}=n\int_{h_0}^{h_1}\left[pq^{+}\frac{D}{q^{+}}\right]f(h_r)\,dh_r-\Omega=pnD-\Omega \quad \cdots\cdots\cdots\cdots (3.9)$$

where q^{+}=economic order quantity of the retailer with the fixe unit price schedule

$$=\left\{\frac{2K_rD}{h_r}\right\}^{1/2} \quad \cdots\cdots\cdots\cdots (3.10)$$

Now, the economic order quantity of the retailer with the new pricing policy can be determined from the retailer's self-selection condition. From Eq.(2.2), we get

$$q^{*}=\left\{\frac{2K_rD}{\dfrac{F(h_r)}{f(h_r)}+h_r}\right\}^{1/2} \quad \cdots\cdots\cdots\cdots (3.11)$$

Then, the wholesaler's expected net profit under the new pricing schedule is as below.

$$\pi_{new}=n\int_{h_0}^{h_1}\Big[\Big\{-K_r-\frac{q}{2D}\int h(q)dq+C_0q\Big\}\frac{D}{q}\Big]_{q=q^*} f(h_r)dh_r-\Omega \quad \cdots\cdots (3.12)$$

Thus, wholesaler's expected net profit condition $\pi_{new}\geq\pi_{old}$ implies

$$C_0\geq p+\int_{h_0}^{h_1}\Big[\frac{K_r}{q}+\frac{1}{2D}\int h(q)dq\Big]_{q=q^*} f(h_r)dh_r \quad \cdots\cdots (3.13)$$

For the convenience of notation, let $\int h(q)dq=H(q)$. Then we will get the necessary condition for C_0 which guarantees an increase in the wholesaler's profit as

$$C_0\geq p+\int_{h_0}^{h_1}\Big\{\frac{K_r}{q^*}+\frac{1}{2D}H(q^*)\Big\}f(h_r)dh_r=C_{min} \quad \cdots\cdots (3.14)$$

3.2. Retailer's Total cost.

The ratailer's total cost is composed of the purchasing cost, ordering cost and inventory holding cost. The retailer's total cost with the old pricing policy and with new pricing policy can be expressed as follows.

$$TC_{old}=pD+K_r\frac{D}{q^+}+\frac{q^+}{2}h_r=pD+(2DK_rh_r)^{1/2} \quad \cdots\cdots (3.15)$$

$$TC_{new}=p(q)\frac{D}{q}+K_r\frac{D}{q}+\frac{q}{2}h_r \Big|_{q=q^*}$$

$$=-\frac{1}{2}H(q^*)+C_0D+\frac{q^*}{2}h_r \quad \cdots\cdots (3.16)$$

Consequently, the retailer's total cost condition $TC_{new}\leq TC_{old}$ implies

$$C_0\leq p+\Big\{\frac{2K_rh_r}{D}\Big\}^{1/2}+\frac{1}{2D}H(q^*)-\frac{q^*}{2D}h_r$$

$$=C_{max}(h_r) \quad \cdots\cdots (3.17)$$

If is was possible to get C_0 satisfying Eqs. (3.14) and(3.17), we would get the non-linear pricing schedule that benefits both parties.

3.3. Existence of a Pareto Superior Pricing Schedule.

Under cettain circumstances, there always exists a C_0 satisfying both Eqs.(3.14) and(3.17). But under some other conditions, no C_0 satisfies both conditions. In order to investigate the circumstances in which C_0 leads to Pareto superior pricing policy, we must take a close look at the behavior of the $C_{max}(h_r)$ as the retailer's unit holding cost changes.

$$\frac{d}{dh_r}C_{max}(h_r)=\frac{1}{2D}(q^+-q^*)$$

Let's assume that indeed the new pricing policy benefits every retailers. Then, because $C_{max}(h_r)$ is an increasing function of h_r, the largest possible value of C_0 should be $C_{max}(h_0)$. But, from Eq.(3.11) the old economic order quantity is exactly the same as the new economic order quantity at the minimum value of the unit holding cost range or the maximum retailer's ordering quantity. And, From Eq.(3.8), our new pricing schedule has negative value on P(q) axis.

From these facts, we know that the old pricing schedule is a tangent line on the new pricing schedule at the retailer's ordering quantity of $q_1 = q^+(h_0) = q^*(h_0) = \{\frac{2k_rD}{h_0}\}^{1/2}$ as shown in Fig. 1. The new economic order quantity tends to be larger than the old economic order quantity when the marginal price is less than the unit fixed price; on the other hand new economic order quantity tends to be smaller when the marginal price is greater than the fixed unit price.

From Fig. 1, it is evident that the average unit price in the new pricing schedule described above is always below the old fixed unit price for the whole range of the retailer's order quantity. Consequently, the wholesaler's profit will be lower than the fixed unit price schedule and therefore, it is impossible to get a Pareto superior quantity discount pricing schedule in the multiple retailers case.

The only possible case in which the wholesaler as well as all the retailers can gain from the new pricing schedule is when all the retailers have the same inventory holding cost h_0. But this case reduces to the case of one wholesaler in which the wholesaler has perfect information on the retailer's characteristics[Monahan 1984, Rosenblatt and Lee 1985, 1986].

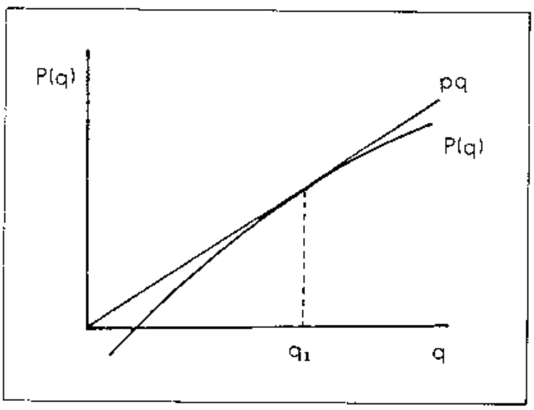

Fig. 1. Shape of old and new pricing schedule.

4. Conclusion.

So far, we have discussed the non-linear pricing schedule with only one underlying retailer's characteristics from the viewpoint of a wholesaler. Using the same kind of analysis, we can get an idea of the behavior of the retailer's new economic order quantity and the behavior of the wholesaler's expected profit when there are two or more underlying retailer's characteristics in the case of multiple ratailers.

When we analyze the model using only the information on the probability distribution function of the retailer's ordering cost or demand rate, the retailer's new economic order quantity has the term $\frac{F(\cdot)}{f(\cdot)}$ in the numerator without the $\frac{F(h_r)}{f(h_r)}$ term in the denominator. Examining the wholesaler's expected profit function, we already know that expected profit comes from expected gross revenue and constant cost terms. By the same reasoning as in the heterogeneous unit inventory holding cost, we can say there is no Pareto superior pricing schedule for these either case.

Eventually, if we consider the retailer's two or more underlying characteristics in the multiple retailers case, we see that it is impossible to get a Pareto superior pricing schedule for the wholesaler except when all the retailers are homogeneous. But this is the same case as the case of one wholesaler who has the perfect information about his only one customer[Monahan 1984, Rosenblatt and Lee 1985, 1986].

In this paper, we dont't consider the shortage case. If shortages are allowed on either side, then the economic order quantity of the wholesaler can be a function of the retailer's economic order quantity. Moreover, it is practical to consider the dynamic change of ordering quantity with discount. All of these relaxation of our assumption suggests the future research area.

References

1. Eppen, Gary D. and Liebermann, Yehoshua: "Why do retailers deal? An Inventory Explanation", *Journal of Business*, 57, 4, 1984.
2. Garbor, A., "A Note on Block Tariffs": *Review of Economic Studies*, 23, 1955.
3. Goldman, M. Barry, Leland, Hayne E. and Sibley, David S.: "Optimal Non-uniform Prices", *Review of Economic Studies*, LI, 1984.
4. Goyal, S.K.: "An Integrated Inventory Model for a Single Supplier Single Customer Problem", *International Journal of Production Research*, 15, 1, 1976.
5. Kamien, Morton I. and Schwartz, Nancy L.: *Dynamic Optimization; The Calculus of Variations and Optimal Control in Economics and Management*, 4,. North. Holland.
6. Kap H. Kim and Hark Hwang.: "An incremental discount pricing schedule with multiple customers and single price break", *European Journal of Operational Research*, 35, 1988.
7. Lal, Rajiv and Staelin, Richard.: "An Approach for Developing an Optimal Discount Pricing Policy", *Management Science*, 30, 12, Dec. 1984.
8. Lee, Hau L. and Rosenblatt, Meir J.: "A Generalized Quantity Discount Pricing Model to Increase Supplier's Profit", *Management Science*, 32, 9, Sept. 1986.
9. Monahan, James P.: "A Quantity Discount Pricing Model to Increase Vendor Profits", *Management Science*, 30, 6, June 1984.
10. Oren, Shmuel S., Smith, Stephen A. and Wilson, Robert B.: "Competitive Nonlinear Tariffs", *Journal of Economic Theory*, 29, 1, Feb. 1983.
11. Oren, Shmuel S., Smith, Stephen A. and Wilson, Robert B.: "Nonlinear Pricing in Markets with Interdependent Demand", *Marketing Science*, 1, 3(Summer). 1982.
12. Rosenblatt, Meir J. and Lee, Hau L.: "Improving Profitability with Quantity Discounts under Fixed Demand", *IIE Transactions*, 17, 4, 1985.