A Competitive Equilibrium Model of the Market for Used Goods

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Abstract

The present paper determines the equilibrium price function of used goods and their carry-over age when there are heterogeneous firms with different factor prices. It is shown that the used good market enables more efficient use of durable goods and thereby gains from trades. It is also shown that firms with a lower interest rate and a higher wage rate specialize in using newer goods.

1. Introduction

Economic analysis of the theory of market equilibrium has traditionally concentrated on markets for nondurable goods. And, it has not paid much attention to the working of the markets for durable goods that explicitly takes account of the structural difference between two types of markets. A market for used goods is rarely assumed to exist; even when it is allowed to exist, it is accorded secondary importance and treated as a market induced from the new good market. But, for example, it is well known that considerable portion of automobile trades involves used cars. Also, small firms have a relatively large portion of used machines in their capital stocks compared to big firms(cf. Shinohara[6]).

Many of the previous works on the market for used goods are not appropriate because of their typical assumption of homogeneous agents. For example, Parks [3], in his study of estimating the used car

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price, assumes that agents in the market are identical and look for a price of used cars at which they are indifferent between new and used cars. However, in reality this is not the case; some people in the market prefer new cars while others prefer used ones so that those two kinds of cars are not perfect substitutes. This in turn implies that more than one type of consumers exist, each having a different preference structure and/or different income.

Bond[1] noticing this imperfect substitutability constructs a model of used equipment trades between heterogeneous firms, differing in their factor prices and utilization rate of the equipment. Although the model explains the working of the used good market in an elegant way, it is too simple to capture some of fundamental implications involved. For example, in his model the capital goods are assumed to last two periods, new in the first period and used in the second period. When the goods ages continuously, the model fails to determine the age at which the goods are transfered from one type of firms to another type.

The purpose of this paper is to develop a "heterogeneous" competitive euqilibrium model of a durable good market in a formal way by explicitly introducing different (two) types of agents competig for capital goods for different ages[1], each agent has a different marginal valuation of the durable good, which is given as a function of age of the good. As a result, as the good ages, it is sold by one agent who places the highest value on it. Specifically, we consider a competitive industry(A) producing a durable good(a machine) that is sold to firms in a second competitive industry(B). Industry B uses the durable good in production, and it is the behavior of firms in this industry that is the main object of study. The final output is sold to consumers. Each firm evaluates a machine of each age by its discounted sum of future quasi-rents based on its own factor prices, and heterogeneity enters because different types of firms face different wages and interest rates[2]. Given this setting, we will determine the equilibrium price function and the carry-over age of the machines.

Section 2 sets up the basic model for subsequent analysis. The main results are in Section 3, with all formal proofs relegated to Appendix. The last section summarizes the results and offers some future research topics.

2. The Model

Consider a model of two competitive

1) Sen[5] and Smith[7] consider a similar problem of international trades used machnies between two countries having different factor prices.

2) The assumption of factor price differentials has compelling empirical evidence. For example, Oi [2] finds that hourly wages in the largest firms are 24.7 percent above wages in the smallest firms, after adjusting the effects of worker characteristic(sex, race, education, job tenure, etc.). Also, Sherer et al.[4] report that there exists a negative correlation between interest cost and firm size.

industries(Industry A and Industry B). Industry A produces machinary and sells them to Industry B. Then Industry B uses those machines to produce final output y, which is in turn sold to consumers. Industry B consists of two types of competitive firms having different cost conditions. A type i(i=1, 2) firm has a wage rate ω_i and an interest rate γ_i, which are assumed to be constant over time. The machine is of one-hoss-shay type. That is, it lives for n years with unchanged efficiency. At each instant of its lifetime, it is used with one unit of labor and produces at the rate of one unit of the final output per unit time. Throughout section 2 and 3, we confine ourselves to a stationary state where the stock of machinary in Industry B remains constant so that the output flow and its price P_y, are constant. Now, suppose that type i firm is operating a machine of age τ with durability n.

The quasi-rent yielded by the machine, $q_i(\tau)$, $(0 \le \tau \le n)$, is defined by

$$q_i(\tau) = q_i = P_y - \omega_i \text{ for } 0 \le \tau \le n. \quad \cdots\cdots (1)$$

Then, it is natural that the frim determines the value of a machine of age t by the discounted sum of future quasi-rent(DSFQ), $P_i(t)$, given by

$$P_i(t) = \int_t^n q_i \exp\{-\gamma_i(\tau - t)\}d\tau$$
$$= q_i(1 - \exp\{-\gamma_i(n-t)\})/\gamma_i \cdots\cdots (2)$$

where $0 \le t \le n$. In particular,

$$P_i(0) = q_i(1 - \exp\{-\gamma_i n\})/\gamma_i$$
$$P_i(n) = 0. \quad \cdots\cdots (3)$$

Differentiating equation(2), we get

$$dP_i(t)/dt = \gamma_i P_i(t) - q_i$$
$$= -q_i \exp\{-\gamma_i(n-t)\} < 0$$
$$d^2P_i(t)/dt^2 = \gamma_i P_i(t) < 0 \text{ for all t.} \quad \cdots\cdots (4)$$

Therefore, P_i is a strictly decreasing and concave function of t. A useful relationship follows from the definition of DSFQ:

$$-P_i(t) + P_i(t') \exp\{-\gamma_i(t'-t)\}$$
$$+ \int_t^{t'} q_i \exp\{-\gamma_i(\tau - t)\}d\tau = 0$$
$$\text{for any t and t}',\ 0 \le t < t' \le n. \quad \cdots\cdots (5)$$

The left hand side of equation(5) gives the profit to the frims associated with purchasing one machine of age t, using it for t′-t years and selling it at age t′, given $P_i(t)$ as the price function.

Then the equation says that a type i firm will collect zero profit no matter what machines it owns if $P_i(t)$ is the price function. Now, let $P^*(t)$ be a price function of machines in the market. Suppose that a type i firm buys one t year old machine, operates it for t′-t years and then sells it(t′>t). Then profits from the transaction are, analogously to equation(5),

$$\pi_i(t,\ t' : P^*) = -P^*(t) + P^*(t') \exp\{-\gamma_i(t'-t)\}$$
$$+ \int_t^{t'} q_i \exp\{-\gamma_i(\tau - t)\}d\tau \quad \cdots\cdots (6)$$

where $0 \le t \le t' \le n$. Since we are considering a competitive market, it is reasonable to restrict the analysis to price functions yielding nonpositive profits. More formally, we only consider the price functions belonging to a set

$\Omega \equiv \{P^*(t) \mid P^* \geq 0$ and is differentiable on $[0, n]$ and $\pi_i(t, t' : P^*) \leq 0$ for all i and $0 \leq t \leq t' \leq n.\}$

Note that in a competitive and stationary situation, each type i firm is concerned only with choosing the t and t', which can give zero(the maximum) profit.

Now, we present the following two useful results which are readily derived from equation(6). First, notice that $P_i(t)$-$P^*(t)$ is the profit of a type i frim when it buys a t year old machine and keeps it until it dies because it is the difference between the discounted value of the machine and the actual price. Suppose that the firm sells the machine when it becomes t' years old($t' > t$). Then it will forgo potential profits($P_i(t')$-$P^*(t')$) $\exp\{-\gamma_i(t'-t)\}$. Therefore, $\pi_i(t,t' : P^*)$ can be rewritten as

$$\pi_i(t, t' : P^*) = P_i(t) - P^*(t) - (P_i(t') - P^*(t'))\exp\{-\gamma_i(t'-t)\}$$
$$= (H_i(t') - H_i(t))\exp\{\gamma_i t\} \cdots (7)$$

where $H_i(\tau) = (P^*(\tau) - P_i(\tau))\exp\{-\gamma_i \tau\}$. Equation(7) is useful in proving the existence of an equilibrium price function since, for example, in order to check if $\pi_i(t,t' : P^*) \leq 0$, we only have to look at the curvature of the function H.

Second, consider a transaction in which a type i firm buys a t year old machine, sells it when it is t'' year old and at the same time buys another t'' year old machine and sells it after t'-t'' years($t < t'' < t'$). This transaction is clearly equivalent to the transaction discussed above. As a result,

$$\pi_i(t, t' : P^*) = \pi(t, t'' : P^*) + \pi_i(t'', t' : P^*)\exp\{-\gamma_i(t''-t)\} \cdots\cdots (8)$$

To see the usefulness of equation(8), consider a P^* in Ω such that $\pi_i(t,t' : P^*) = 0$ for some t and t'. Then the above equation shows that for all t'', $t < t'' < t'$, $\pi_i(t'', t' : P^*) = 0$ for the nonpositivity of profits. (Also, $\pi_i(t, t'' : P^*) = 0$ for all t'', $t < t'' < t'$) In other words, machines of ages between t and t' are equally attractive to the type i firm.

3. Determination of Equilibrium Machine Prices: Used Machine Market

Define an age set $S_i(P^*)$ for $P^* \in \Omega$ by the following:

$$S_i(P^*) = \{t : 0 \leq t < n, \ \pi_i(t, t' : P^*) = 0 \text{ for some } t' > t\} \cdots\cdots (9)$$

Since a machine of an age in $S_i(P^*)$ gives zero(the maximum) profit to type i firm, we interpret the set as a set of machines which is held by a type i firm. Now, the definition of $S_i(P^*)$ and the remark on equal attractiveness in the previous section imply that $S_i(P^*)$ is a half open interval like $[t_1, t_2)$ or a collection of such intervals, which are disconnected[3]. With this preliminary result, we define the competitive equilibrium as follows:

Definition: $P^*(t)$ is the competitive equilibrium price function if(i) $P^* \in \Omega$, (ii) $S_i(P^*) \neq 0$, $(i=1, 2)$, (iii) $U^2_{i=1} S_i(P^*) = (0, n)$.

Condition(i) indicates that no firm can make positive profits from any machine. Condition(ii) implies that all types of firms are active in Industry B and lastly condition(iii) requires that any type of machine should be operated by some firms.

First, consider a(trivial) case where all firms are identical(type 1). Then, condition (ii) and (iii) become a single condition that $S_1(P^*) = (0, n)$ or for all t, there exists $t' > t$ such that $\pi_1(t, t': P^*) = 0$. Since $\pi_1(t, t': P_1) = 0$ for all $t' > t$ as seen in the previous section, $S_1(P_1) = [0, n)$. Obviously, $P_1 \in \Omega$. Therefore, in this trivial case, $P_1(t)$ is the competitive equilibrium price function. However, if there are two types of firms in the market, the situation is not so simple. Suppose that P_1 and P_2 are given in Fig. 1. One may intuitively argue that type 1 firm buys a new machine and operaters it until age T, the age when $P_1(T) = P_2(T)$. Then type 2 firm buys the machine and operates it to a n. It is easy to see that profits are zero for both types of firms for all $t \in [0, n)$ if P^* is given by the upper envelope of P_1 and P_2 and the machine is carried over at age T. However, a problem with this solution is that P^* cannot belong to Ω because a type 1 firm can make positive profirts by buying a T year old machine and selling it after T′-T years. Therefore this intuitive solution is wrong[4]. From this observation, we may raise two related questions: How do we find the competitive equilibrium price function in the heterogeneous firm case? Can we express it in terms of P_1 and P_2 as we did in the case of homogeneous firms? We will explore these questions below.

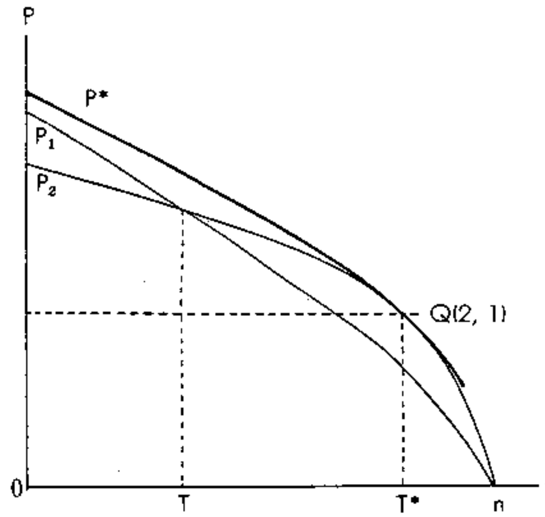

Fig. 1.

3) Note that t_1 is included because a machine is bought at age t_1. Similarly, t_2 is excluded because the machine is sold at age t_2. This can be proved formally as follows: If $t_2 \in S_i$, then there exists $t > t_2$ such that $\pi_i(t_2, t: P^*) = 0$. But we already know that by the equal attrativeness for any $t' \in [t_2, t)$, $\pi_i(t', t: P^*) = 0$ and therefore, $[t_2, t) \subset S_i$, which is contradictory to the fact that$[t_1, t_2)$ is disconnected with other intervals. For the inclusion of t_1, first note that for any t, $t_1 < t < t_2$, $\pi_i(t, t_2) = 0$ by the above argument and equation(9). Then, $0 = \lim_{t \to t_1} \pi_i(t, t_2: P^*) = \pi_i(t_1, t_2: P^*)$ where the second equality follows from the continuity of π_i. This proves the desired result.

4) Another uneasy point about this intuitive solution is that P^* is not differentiable at T.

To begin with, note that the definition of the competitive quilibrium in the previous subsection allows for some ambiguities. For example, S_1 and S_2 may overlap each other so that it is not clear who will own the machines of the overlapped ages. Also, S_i may be disconnected, that is, a type i firm may repurchase a machine which was sold by the firm itself before. However, with our one-hoss-shay assumption, all these ambiguities are shown to disappear. Proposition 1 gives the necessary and sufficient conditions for the competitive euqilibrium price function, and shows that there is a unique age $T^* \in (0, n)$ such that $S_1 = [0, T^*)$ and $S_2 = [T^*, n)$. T^* is called the carry-over age of machine from type 1 firms to type 2 firms; at that age both types of firm are willing to trade the machine. For notational convenience, we define $Q(j, i) = (q_j - q_i)/(\gamma_j - \gamma_i)$ where $j > i$.

Proposition 1: Suppose that $\gamma_2 > \gamma_1$. Then, $P^*(t)$ is the competitive equilibrium price function and $S_i(P^*)$, $(i=1, 2)$ is the corresponding age set if and only if there exists a unique T^*, $0 < T^* < n$ such that

(i) $S_1(P^*) = [0, T^*)$
$S_2(P^*) = [T^*, n)$

(ii) $P^*(t) = P_1(t) + \alpha^* \exp\{\gamma_1 t\}$ for $t \in S_1(P^*)$
$= P_2(t)$ for $t \in S_2(P^*)$

(iii) $P^*(t) > Q(2, 1)$ for $t \in S_1(P^*)$
$P^*(t) \leq Q(2, 1)$ for $t \in S_2(P^*)$

where $\alpha^* = (P_2(T^*) - P_1(T^*)) \exp\{-\gamma_1 T^*\} > 0$ and in (iii) the strict inequality holds only at $t = T^*$.

As mentioned above, condition(i) exludes any possible ambiguities. Moreover, we have an important result that in the competitive equilibrium(if it exists), firms with a lower interest rate(type 1 firms in this case) use newer machines and vice versa. Next, we give some economic content to the rather mathematical conditions (i) and (iii). For condition (ii), we define first the asset value of a machine of age when it is used by a type i firm for the first T years and by a type j firm for the remaining years, $V(t: T)$, as follows: Let $\alpha = (P_j(T) - P_i(T)) \exp\{-\gamma_i T\}$.

$$V(t:T) = \int_t^T q_i \exp\{-\gamma_i(\tau - t)\} d\tau + \exp\{-\gamma_i(T-t)\} \int_T^n q_i \exp\{-\gamma_j(\tau - t)\} d\tau = P_i(t) + \alpha \exp\{\gamma_i t\} \text{ for } 0 \leq t \leq T,$$

$$V(t:T) = \int_t^n q_i \exp\{-\gamma_j(\tau - t)\} d\tau = P_j(t) \text{ for } T \leq t. \quad \cdots\cdots\cdots\cdots (10)$$

Condition (ii) in Proposition 1 then states that, in equilibrium, the market valuation of a machine must be equal to its asset value defined above[5].

Next, the cost of operating a machine of age t by a type i firm is a sum of the capital cost, the wage cost and the price depreciation, i.e.,

5) Note that our earlier trial solution also fits this condition because if we take T as a carry-over age in Fig. 1, then $\alpha = 0$ so that $v(t:T) = P_1(t)$ for $t < T$ and $v(t:T) = P_2(t)$ for $t \geq T$. A problem with this solution is that it does not satisfy condition (iii) because of the wrong choice of the carry-over age.

$\gamma_1 P^* + \omega_1 - dP^*/dt.$ ························ (11)

Then, $(\gamma_1-\gamma_2)P^*+(\omega_1-\omega_2)=(\gamma_1-\gamma_2)P^*-(q_1-q_2)$ is the comparative advantage of a type 1 firm in operating the machine. Therefore, recalling the definition of Q(2, 1), condition (iii) implies that, in equilibrium, a firm uses machines for which it has the comparative advantage. Note that the comparative advantage is a sum of comparative advantage in terms of capital cost$(\gamma_1-\gamma_2)P^*$ and labor costs $-(q_1-q_2)$. The condition can be interpreted in a different way. As an illustration, pick type 1. For $t \in S_1(P^*)$, $\gamma_1 P^*+q_1-dP^*/dt=P_y$, using condition (ii). Since the machine produces one unit of the final output, the firm makes zero instantaneous profits from the machine. However, for $t \in S_2(P^*)$, $\gamma_1 P^*+\omega_1-dP^*/dt=(\gamma_1-\gamma_2)P^*+(\omega_1-\omega_2)+P_y=(\gamma_1-\gamma_2)P^*-(q_1-q_2)+P_y>P_y$.

Therefore, the firm will obtain negative instantaneous profits from owning the machine. In other words, for a machine to be held, it must give a zero rate of instantaneous profits.

Even though the proposition consists of threee sets of conditions to solve for three sets of unknowns. T^*, P^* and S_1, it is not yet straightforward to find T^* and hence P^*. In the following, we present a simple rule for finding T^* in the next proposition, giving a clue on how construct the competitive equilibrium price function.

Proposition 2: Suppose that $\gamma_2>\gamma_1$. Then, for the competitive equilibrium, it is necessary that

$$P_2(0)>Q(2, 1)>0.$$

Or equivalently ther exists a unique T^* such that

$$P_2(T^*)=Q(2,\ 1),\quad 0<T^*<n.$$

Also this is sufficient for the competitive equilibrium. In particular, if the above equation holds, (i) $(P_2(t)-P_1(t))\exp\{-\gamma_1 t\}$ is maximized at T^*. Let α^* be its maximized value. Then $\alpha^*>0$. (ii) Identifying T^* in the above equation as that in Proposition 1, the competitive equilibrium exists and the competitive euqilibrium price function and the corresponding age sets are given by conditions (ii) and (i) in Proposition 1, respectively. (iii) $P_1(t)+\alpha^*\exp\{\gamma_1 t\}$ is tangent to $P_2(t)$ at T^*. (iv) $dP^*(t)/dt<0$ and $d^2P^*(t)/dt^2<0$ for all t, $0\le t\le n$.

The first part of the proposition implies that if $\gamma_2>\gamma_1$, then $q_2>q_1$ or $\omega_1>\omega_2$ in the competitive equilibrium. That is, for the used market to be operating, one type should not dominate the other in terms of cost conditions, which is clearly in accord with intuition. To see the meaning of result (i), suppose that initially only type 1 firms exist in the market so that $P^*(t)=P_1(t)$. Furthermore assume that $P_1(0)=C$ so that firms in Industry A make zero profit where C is the constant marginal cost of firms in Industry A. Now, suppose that type 2 firms enter the market so that type 1 firm can sell its machine of age t to a type 2 firm. Then the profits the type 1 firm can get are $\pi_1(0,\ t:\ P_1)=-C+P_2(t)\ \exp\{-\gamma_1 t\}+\int_0^t q_1 \exp\{-\gamma_1\tau\}d\tau$.

Subtracting the identity $0=-P_1(0)+P_1(t)$

$\exp\{-\gamma_1 t\} + \int_0^t q_1 \exp\{-\gamma_1 \tau\} d\tau$ gives $\pi_1(0, t) = (P_2(t) - P_1(t)) \exp\{-\gamma_1 t\}$[6]. Then result (i) says that the machine is transferred at an age which maximizes $\pi_1(0, t)$. Since $\pi_1(0, T^*) = \alpha^* > 0$, there will be competition among type 1 firms for acquiring the machine of age T^*, raising its price up to $P_2(T^*)$ so that no extra profits are possible. And in the process, prices of all other machines will rise enough to rule the possible arbitrage.

Now, we will look at a way of constructing the equilibrium price function using results (ii) and (iii). First draw $P_2(t)$ and a horizontal line passing through Q(2, 1). Denote an intersection point by T^*, which is the unique carry-over age from type 1 firms to type 2 firms. Next, draw $P_1 + \alpha^* \exp\{\gamma_1 t\}$ which is tangent to P_2 at T^* by result (iii). Then, P^* can be obtained by taking $P_1 + \alpha^* \exp\{\gamma_1 t\}$ for $t < T^*$ and P_2 for $t \geq T^*$ as the thick line in Fig. 1. Finally, result(iv) shows that P^* is well-behaved in that it is differentiable, decreasing and concave like P_1.

Some remarks on the welfare aspects are in order. We have seen that if type 2 firms are added to the market where originally only type 1 firms operate, the equilibrium price function shifts upward from P_1 to P^*. This increase in prices reflects the more efficient use of machines caused by the introduction of another type of firms. In other words, as type 2 firms, which are more efficient in the operation of used machines, specialize in the use of those machines, the efficiency of machines for society in general is raised. That is, the establishment of the used machine market generates the gains from trades.

Furthermore, the machine is carried over at the age which maximizes the asset value of machines. For example, even when the machine is carried over at T in Fig. 1, there is an increase in the efficiency of the use of the machines which raises their asset value, particularly for the machines older than T Years. However, it is obvious that the gains are dominated by what are obtained if the carry-over occurs at T^*. In this sense, T^* is the optimal carry-over age. A formal statement of this observation is given by the following proposition.

Proposition 3: Given the situation in Propositions 2, for any $T \neq T^*$, $V(t:T^*) \geq V(t:T)$ for all t and strict inequality holds for some t where T is the carry-over age either from type 1 to type 2 or the other way around.

4. Summary and Extension

We have constructed the equilibrium price function from the DSFQ's, and found it preserves their properties. Contrary to intuition, the carry-over age of a machine is not an intersection points of the DSFQ's, where the absolute efficiency of each type is equal. By contrast, it is determined by the relative efficiency of the firms of the comparative advantage. Naturally, gains acc-

6) This can be derived from equation(7).

rue from the trades of machines.

A result worth noticing is that firms with a lower interest rate and higher wage rate will specialize in using newer machines. From the discussions in Section 3, it is clear that this proposition continues to hold for neoclassical production Function where the quasi-rents are a function of age, as long as $(q_2(t)-q_1(t))/(\gamma_2-\gamma_1)$ cuts the DSFQ of type 2 firms from below. However, if the production function has a vey high elasticity of substitution so that a small change in the wage rate results in a large difference in the quasi-rent stream in the earlier age machines, the result may be reversed. In other words, $(q_2(t)-q_1(t))/(\gamma_2-\gamma_1)$ may cut the DS-FQ of type 2 firms from above. In this case, the comparative advantage of a firm with a lower wage rate and a higher interest rate(i. e., a type 2 firm) in terms of labor cost may exceed its comparative disadvantage in terms of capital cost for machines of the early ages. Then, type 2 firms will use the newer machines.

Heterogeneity in the evaluation of machines is generated by factor price differentials in this analysis. Of course, there may be other sources produciing such heterogeneity. For example, firms may have different production functions, resulting in different evaluations. Or firms having easier access to technical innovations will generally put less value on used machines than otherwise. It will be interesting to investigate those other sources, even though the basic results seem likely to remain intact.

Finally, although we present the model of two types of firms, it is almost straightforward to extend it to the case of many types of firms. In the latter case of potentially many types of firms in the market, it will be interesting to investigate a problem of picking firm types that can actively engage in the used good trade.

Appendix

1. Proof of Proposition 1

Necessity: First we note that $P^*(n)=0$ so that $H_1(n)=0$ for nonpositive profits where H_1 is defined in equation (7). Next, we observe that from the definition the competitive equilibrium price function, either of the following must hold.

Case 1: There exists $T^*<n$ such that $[T^*,n)\subset S_2$ and for any sufficiently small $\delta>0$, $T^*-\delta\in S_1$.

Case 2: There exists $T^*<n$ such that $[T^*,n)\subset S_1$ and for any sufficiently small $\delta>0$, $T^*-\delta\in S_2$.

Our strat﹏y is that we prove the proposition by assuming that Case 1 holds and next, using the result obtained, show that Case 2 is in fact impossible to hold under the assumption that $\gamma_2>\gamma_1$ given in Proposition 1. Suppose that Case 1 holds. From the definition of H_2 and

$H_1(n)=0$, $P^*=P_2$ for $t\in[T^*, n)$. Similarly, there must exist an interval$[T_1, T^*)\subset S_1$ where $P^*=P_1+\alpha^*\exp\{\gamma_1 t\}$ for some constant $\alpha^*=(P_2(T^*)-P_1(T^*))\ \exp\{-\gamma_1 T^*\}$(for continuity of P^* at T^*). Now $dP^{*+}(T^*)/dt=$

$\gamma_2 P^*(T^*)-q_2$ and $dP^{*-}(T^*)/dt=\gamma_1 P^*(T^*)-q_1$ where superscripts + and - atatched to P^* indicate right- and left- derivatives, respectively. Since P^* is differentiable everywhere, we get

It is obvious that no $t>T^*$ can satisfy the above equation because $P^*=P_2$ for $t\in[T^*, n)$, which is strictly decreasing. This implies that $S_1\cap[T^*, n)=0$ because otherwise t in the set satisifes the condition in Case 1 so that it must satisfy the above equation, which was shown above to be impossible. Now suppose that there is $t<T^*$ at which $P^*(t)=Q(2, 1)$. Then, from the above equation, $\pi_2(t, T^*:P^*)=-H_2(t)\exp\{\gamma_2 t\}>0$ because $P_2(t)>P_2(T^*)=Q(2,1)=P^*(t)$, which is a contradiction. Therefore, T^* is the unique age at which equation the above equation holds.

Now, we show that $S_2=[T^*, n)$. If $[T^*, n)$ is a proper subset of S_2, there exists $T'<T^*$ such that $T'\in S_1$ and for any sufficiently small $\delta>0$, $T'-\delta\in S_2$.

Applying the same argument as before, we have $P^*(T')=Q(2, 1)$, which is impossible. Therefore, $S_2=[T^*, n)$ and as a result $S_1=[0, T^*)$, which proves (i). (ii) follows immediately from the previous discussion except $\alpha^*>0$, which is proved by noting that $\pi_1(T^*n:P^*)=-\alpha^*\exp\{\gamma_1 T^*\}<0$ because $T^*\notin S_1$. For a proof of(iii), note that if $t\in S_1$, $\pi_2(t, t':P^*)<0$ for all $t'>t$, which implies that $H_2(T)<0$ for $t\in S_1$. Since $H_2=(P^*-P_2-\gamma_2(P^*-P_2))\exp\{-\gamma_2 t\}=(\gamma_2-\gamma_1)(-P^*+Q(2, 1))\exp\{-\gamma t\}<0$(the second equality follows from $P^*=\gamma_1 P^*-q_1$ for $t\in S_1$), the first inequality in(iii) obtains immediately. The second inequality also derives analgously.

Next, we have to prove that Case 2 is in fact impossible if $\gamma_2>\gamma_1$. For this, assuming that it holds and following the same argument as before, we end up with a variation of(iii) as follows. $(\gamma_1-\gamma_2)P_1(t)<q_1-q_2$ for $t\in S_1=[T^*, n)$. But $\gamma_1-\gamma_2<0$ so that inequality is rearranged to give $P_1(t)>Q(2, 1)$ for $t\in S_1$. This is contradictory because P_1 is strictly decreasing.

Sufficiency: First note that P^* is differentiable everywhere, in particular, at T^*. Moreover, $S_i\neq 0(i=1, 2)$ and $S_1\cup S_2[0, n)$. Therefore, it suffices to show that for $t\in S_i(P^*)$, there exists $t'>t$ such that $\pi_i(t, t':P^*)=0$ and for $t\in S_j(P^*)$, $\pi_i(t, t':P^*)<0$ for any $t'>t$, $(i, j=1, 2, i\neq j)$. Equivalently, we show the following: (i) $dH_i(t)/dt=0$ if $t\in S_i$, $i=1, 2$. (ii) $dH_1(t)/dt<0$ if $t\in S_2-\{T^*\}$ and $dH_2(t)/dt<0$ if $t\in S_1$. (i) follows from the fact that $H_1(t)=\alpha^*$ for $t\in S_1$ and $H_2(t)=0$ for $t\in S_2$. For (ii), note that $dH_1(t)/dt=(\gamma_2-\gamma_1)(P^*(t)-Q(2, 1))\exp\{-\gamma_1 t\}<0$ for $t\in S_2-\{T^*\}$. Simmilarly, $dH_2(t)/dt=(\gamma_1-\gamma_2)(P^*(t)-Q(2, 1))\exp\{-\gamma_2 t\}<0$ for $t\in S_1$.

2. Proof of Proposition 2

Necessity: This was already proved in the proof of Proposition 1.

Sufficiency: (i) For the maximum of$(P_2-P_1)\exp\{-\gamma_1 t\}$, differentiate it with respect to t to have $d((P_2-P_1)\exp\{-\gamma_1 t\})/dt=(\gamma_2-\gamma_1)(P_2-Q(2, 1))\exp\{-\gamma_1 t\}=0$, which is realized at T^*. Also, for $t>T^*$, for $t>T^*$, $d((P_2-P_1)\exp\{-\gamma_1 t\})/dt<0$ and, for $t<T^*$, $d((P_2-P_1)\exp\{-\gamma_1 t\})/dt>0$. Therefore, T^* is the global maximum point. Finnally, the positivity of α^* follows if we note that $\alpha^* > \lim_{t\to n}(P_2-P_1)\exp\{-\gamma_1 t\}=0$. (ii) We already know that conditions (i), (ii) (iii) of Proposition 1 are sufficient for the equilibrium. Therefore, we only have to prove condition (iii). Since $P^*=P_2$ for $t\in S_2$, the second inequality is immediate from the definition of T^*. For the first inequality, note that $\alpha^* > (P_2-P_1)\exp\{-\gamma_1 t\}$ for all t(in particular, $t\in S_1$). Then, $P^*=P_1+\alpha^* \exp\{\gamma_1 t\} > P_1+P_2-P_1=P_2$. Again from the definition of T^*, $P^*>P_2>Q(2,1)$. (iii) At $t=T^*$, $d(P_1+\alpha^*\exp\{\gamma_1 t\}-P_2)/dt=\gamma_1(P_1+\alpha^*\exp\{\gamma_1 t\})-\gamma_2 P_2+q_2-q_1=0$ because $P_1(T^*)+\alpha^*\exp\{\gamma_1 T^*\}=P_2(T^*)$ and $(\gamma_1-\gamma_2)P_2+q_2-q_1=0$ at $t=T^*$. Moreover, we already know that $P_1+\alpha^*\exp\{\gamma_1 t\}>P_2$ for all t, which completes proof. (iv) It is obvious that $P^*<0$ for $t\in S_2$. For $t\in S_1$, $dP^*/dt=\gamma_1 P^*-q_1$ and $d^2P^*/dt^2=\gamma_1 P^*$. Since $dP^*(T^*)/dt=dP_2(T^*)/dt<0$, $dP^*/dt<0$ for all $t\in S_1$.(For if there exists $t\in S_1$ such tat $dP^*/dt>0$, there is an interval in S_1 where $dP^*/dt>0$ and $d^2P^*/dt^2<0$, which is a contradiction)

3. Proof of Proposition 3

Proving $V(t{:}T^*)>V(t{:}T)$ is trivial when a machine is carried over from a type 1 firm to a type 2 firm because α is maximized at T^*. For a proof of the other case where the machine is carried over from type 2 firm to a type 1 firm, notice that $V(t{:}T)=P_2(t)+\alpha\exp\{\gamma_2 t\}$ for $t<T$ and $=P_1(t)$ for $t\geq T$ where $\alpha=(P_1(T)-P_2(T))\exp\{-\gamma_2 T\}$. Suppose that P_1 and P_2 do not intersect. In this case, α is maximized at $T=n$ and equals zero.

Thus $V(t{:}n)=P_2(t)\leq V(t{:}T^*)$. Next, suppose that P_1 and P_2 intersect once. Let the intersection point be T'. If $T\in[T', n)$, it is obvious that $V(t{:}T^*)\geq V(t{:}T)$ because $\alpha\leq 0$. If $T\in[0, T')$, α is maximized at 0, which implies that $V(t{:}T)=P_1(t)<V(t{:}T^*)$.

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