An Application of the Aumann–Sharpley Prices for Joint Cost Allocation through Book Profit

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Aumann-Sharpley 가격에 의한 공통 제조원가의 배분 –상대적 이익 기여도를 중심으로–

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Abstract

We study the joint cost allocation based on the book profit producing power of the output through the A-S price mechanism. We show what part of the A-S book profit is allocated to the joint cost and what part is allocated to the variable total book profit of the short-run book profit function. Also we compare some other classical joint cost allocation methods with this A-S price method.

1. Introduction

Joint products arise whenever a single resource produces a diverse set of useful outputs. One of the problems is to allocate the cost of the single resource to the different products that are derived from it. The motivation for devising joint cost allocation arise because of the need for product costing and divisional measurement and control.

Among the many arbitrary allocation methods[Thomas, 1980, 1979], Aumann-Sharpley(A-S) price mechanism is applied to the joint cost allocation in this paper. Usually to compute A-S price only the cost structure and the output vector must be known to allocate the joint cost to each output vector[Mirman, Samet and Tauman, 1984]. If the allocation only depends on the cost function and output vector, the allocation may bear no relationship to the

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profit producing power of the individual output vector[Kaplan, 1982]. In this paper A-S price mechanism is applied through the book profit(hereafter, we use 'profit' in place of 'book profit') producing structure to compute A-S prices(actually it means A-S book profits).

2. The Axiomatic Approach to Joint Cost Allocation

We present a typical joint costing situation for analysis as follows. Joint cost is to be allocated to the joint products which can be acquired separately, either internally or externally. Cost structure of producing joint product is assumed to be known. And we assume that the cost of the least expensive way of buying a batch of completed product separately and externally is also know. To relate cost allocation to the profit producing power of the individual output vector, we need to define profit producing function. We use the term "characteristic function" to typify this profit producing function. Following definitions will be helpful to get the exact idea of characteristic function.

Definitions.

(1) Coalition: Possible combinations of cost objectives which can act jointly(e.g. be produced, buy a service).

(2) Characteristic function: This specifies the "values" of a particular coalition in our case, this will be the total profit that could be earned by the coalition when it arranges its affairs optimally.

We shall determine the profits for those products when α is produced or acquired and sold and the characteristic function is V. V is a characteristic function of m products, for some m, and α is an m-vector of those products. A profit mechanism is a function B which assigns to each pair(V, α), a vector of profits.

$$B(V, \alpha) = (B_1(V, \alpha), B_2(V, \alpha), \cdots\cdots, B_m(V, \alpha))$$

Let's now consider this profit mechanism which satisfies the following axioms which are derived from the set of axioms that are imposed on A-S price mechanism[Mirman, Samet and Tauman, 1984].

Axiom 1. Profit Sharing.

The profits determined by B for the pair (V, α) cover the profit of selling V(α), i.e, $\alpha * B(V, \alpha) = V(\alpha)$.

Axiom 2. Rescaling.

If the scales of measurement of the commodities are changed, then the profit determined by B are changed accordingly.

Let $\lambda * \alpha = (\lambda_1\alpha_1, \lambda_2\alpha_2, \cdots\cdots, \lambda_m\alpha_m)$ and $G(x) = V(\lambda \times x)$ i, e, the characteristic function G and V differ only in the scales of the commodities(λ_i is the scaling factor), then $B(G, \alpha) = \lambda * B(V, \lambda * \alpha)$.

Axiom 3. Consisitency.

Let $V(x_1, x_2, \cdots\cdots, x_m) = G(\sum_{i=1}^{m} x_i)$,
then $B_i(V, \alpha) = B(G, \sum_{i=1}^{m} \alpha_i)$.

This axiom implies that commodities which have the same effect on the profit have the same A-S profits.

Axiom 4. Positivity.

If V is non-decreasing for some α, then $B(V, \alpha) \geq 0$. It asserts that if V increases at least as rapidly as G, then the profits determined for (V, α) are at least as high as those determined for (G, α) i, e, if $V(0) \geq G(0)$ and V- G is non-decreasing then

$B(V, \alpha) \geq B(G, \alpha)$.

Axiom 5. Additivity.

Whenever a characteristic function can be broken into two components V and G, then calculating the profits determined by characteristic function at point α can be accomplished by adding the profit determined by V and G respectively.

$B(V+G, \alpha) = B(v, \alpha) + B(G, \alpha)$

In order to see the existence and uniqueness of a profit mechanism which obeys these five axioms, we now need to know whether ther is fixed components in characteristic function and the continuity of the characteristic function. All coalition values of union of null coalition V(0) must equal zero. From the definition of characteristic function, a coalition value must be maximum that the coalition can produce. If we calculated a particular coalition value to be negative, the coalition could always increases its value to zero by simply withdrawing from the process.

Negative value can't be the maximum output that the coalition can produce. Intuitively, the value of the null coalition is zero. To apply the A-S price mechanism to this profit mechanism, we assume that characteristic function in continuously differentiable.

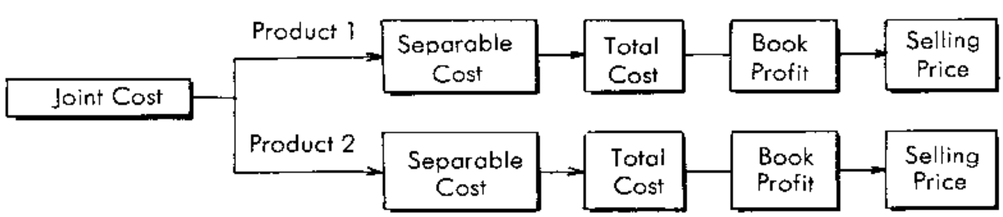
By adopting A-S price mechanism, there exists one and only one profit mechanism B(·, ·) which obeys axiom 1-5[Mirman, Samet and Tauman, 1984].

This is given by

$$B_i(V, \alpha) = \int_1^0 \frac{\partial V}{\partial x_i}(t\alpha)\,dt \quad \cdots\cdots\cdots\cdots (1)$$

3. A-S Book Profit with Joint Cost

We now decide the profit of each output vector. By subtracting these profits from the selling price of each output vector, we can get the total cost of each output. And then by subtracting the separable(traceable) costs associated with output vector, we can decide the joint cost allocation to output vector. This is illustrated as follows.



This is the joint cost allocation method not using the cost function but using the profit function(characteristic function) and get joint cost allocation, which bears the profit producing power of the individual output vector. This approach is applied only to the family of continuous characteristic function V with $V(0)=0$. The long-run characteristic function can be thought of as the envelope of "short-run" characteristic function as Fig. 1. in one dimensional case.

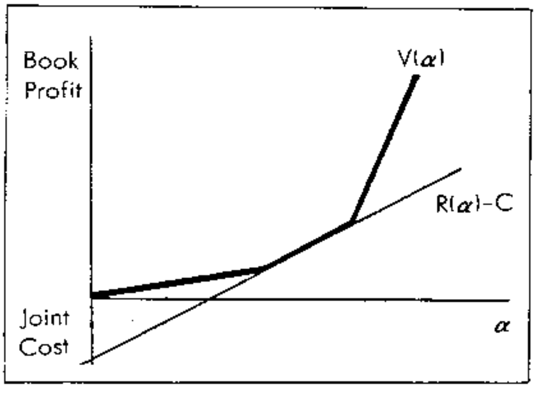


Fig. 1 Long-run characteristic function.

Generally, the short-run characteristic function have a joint cost component as above figure. Let's define by C the joint cost of short-run characteristic function and $R(\alpha)$ the variable book profit of selling α. Then $V(\alpha)=R(\alpha)-C$ at point α. The profit mechanism determines profits for α using the characteristic function $V(\alpha)$ which is composed by both variable book profit $R(\alpha)$ and the joint cost C of selling α.

We want to know what part of these profits is due to the variable book profit $R(\alpha)$ and what part is due to the joint cost C. Since $V(\alpha)=R(\alpha)-C$ and $\sum_{i=1}^{m} B_i\alpha_i=V(\alpha)$, it is possible to calculate another set of profits $\hat{B}$ due only to the variable book profit function R.

$$\hat{B}_i=\int_0^1 \frac{\partial R}{\partial x_i}(t\alpha)\,dt \quad (i=1,\ 2,\cdots\cdots m) \qquad \cdots\cdots (2)$$

Since $\sum_{i=1}^{m} \hat{B}_i\alpha_i=R(\alpha)$, $\sum_{i=1}^{m}(B_i-\hat{B}_i)\alpha_i=V(\alpha)-R(\alpha)=-C$. It means that $\hat{B}_i-B_i$ is that part of book profit B_i, which may be thought of as covering the joint cost C. We think that long-run characteristic function is piecewise linear and all short-run characteristic function are linear. Let H be linear characteristic function of the form.

$$H_j(x_1,\ x_2,\cdots\cdots,x_m)=R_j(x_1,\ x_2,\cdots\cdots,x_m)-C_j=\sum_{i=1}^{m} a_i^j\, x_i - C_j$$

(where $C_1=0$ and a_i^j is the variable book profit of i^{th} good under the range of $t_{j-1}\leq t\leq t_j$)

And these linear pieces are divided into k intervals for producing output vectors $t\alpha$ most efficiently.

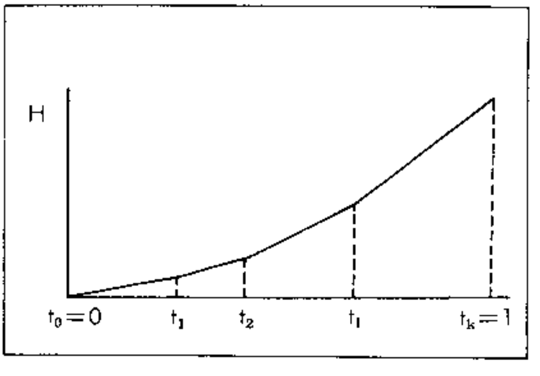


Fig. 2 Long-run piece-wise linear characteristic function.

The A-S profit of i^{th} good determined by (V, α) is given by

$$B_i = \sum_{j=0}^{k-1} (t_{j+1}-t_j) \frac{\partial R_{j+1}}{\partial x_i} \quad \cdots\cdots\cdots\cdots\cdots (3)$$

The part $\hat{B}_i$ of the book profit B_i used to cover the joint cost is the marginal profit of i^{th} good in the range of $t_{k-1} \leq t \leq t_k$.

$$\hat{B}_i = \frac{\partial R_k}{\partial x_i} = a_i^k \ (\hat{B}_i \geq B_i) \quad \cdots\cdots\cdots\cdots\cdots\cdots (4)$$

Therefore $\hat{B}_i - B_i = -\sum_{j=0}^{k-1} (t_{j+1}-t_j) \frac{\partial R_{j+1}}{\partial x_i} + \frac{\partial R_k}{\partial x_i}$ is the part of B_i which covers the joint cost C. The part $\hat{B}_i - B_i$ of profit B_i which is used to cover the joint cost C of selling α under the region of $t_{k-1} \leq t \leq t_k$ depends on the difference between the variable profit function R_k and the long-run characteristic function V for output level which cannot be efficiently produced or acquired and sold by the short-run technology.

(Example 1)

Consider the selling of two goods. Assume there are only two available technologies for selling these goods. Profit functions are given below.

$$H_1(x_1, x_2) = x_1 + x_2 = R_1(x_1, x_2)$$
$$H_2(x_1, x_2) = 3x_1 + 2x_2 - 9 = R_2(x_1, x_2) - 9$$

Since $H_2 \geq H_1$ when $2x_1 + x_2 \geq 9$, the maximum profit is

$$V(x_1, x_2) = 3x_1 + 2x_2 - 9 \text{ if } 2x_1 + x_2 \geq 9$$
$$= x_1 + x_2 \quad \text{otherwise}$$

If $\alpha = (6,6)$ then $V(6,6) = 21$.

Clearly as in Fig. 3. $t_0 = 0$, $t_1 = 1/2$, $t_2 = 1$

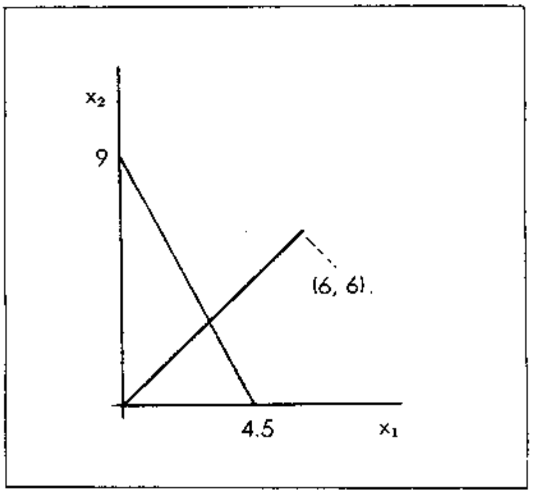


Fig. 3 An example for a piecewise linear function.

$$B_1 = \frac{1}{2}\frac{\partial R_1}{\partial x_1} + \frac{1}{2}\frac{\partial R_2}{\partial x_1} = 2$$
$$B_2 = \frac{1}{2}\frac{\partial R_1}{\partial x_2} + \frac{1}{2}\frac{\partial R_2}{\partial x_2} = 1.5$$
$$\hat{B}_1 = \frac{\partial R_2}{\partial x_1} = 3 \qquad \hat{B}_2 = \frac{\partial R_2}{\partial x_2} = 2$$
$$B_1 - \hat{B}_1 = -1 \qquad B_2 - \hat{B}_2 = -0.5$$

Thus the profit vector(3, 2) covers the variable total book profit of output level(6, 6), 1/3 of the first good and 1/4 of the second good may be attributed to covering the joint cost, i, e, $1 \times 6 + 0.5 \times 6 = 9$(short-run joint cost).

From Eq. 1., A-S profits have the following interpretation. Charge each unit of the first $t_1\alpha_i$ units of i^{th} good the marginal profit of selling this unit $\frac{\partial R_1}{\partial x_i}$, then charge each unit of the next$(t_2-t_1)\alpha_i$ units its marginal profit $\frac{\partial R_2}{\partial x_i}$……, etc. Finally, we can get

$$\alpha_i B_i = t_1 \alpha_i \frac{\partial R_1}{\partial x_i} + (t_2 - t_1)\alpha_i \frac{\partial R_2}{\partial x_i} + \cdots\cdots + (1 - t_{k-1})\alpha_i \frac{\partial R_k}{\partial x_i}$$

Therefore, profit per unit is B_i, the A-S book profit of the i^{th} good.

4. Comparison with Other Joint Cost Allocation Method

For reference it may be useful to introduce some other classical joint cost allocation methods which take account into the profit-producing power of the individual goods.

Let's define

J_i = Joint cost allocation to i (i=subsequent denoting a product)

f_i = Separable cost associated with i.

T_i = Total cost associated with i after the allocation($T_i = J_i + f_i$).

X_i = Least-cost method of acquiring i externally and separately.

Y_i = Least-cost method of acquiring i separately, either internally or externally.

p_i = Revenue generated by i.

n_i = Net realizable value associated with i ($n_i = p_i - f_i$).

b_i = Book profit associated with i($b_i = p_i - T_i$ or $p_i - x_i$).

(1) NRV(Net Realizable Value) method

$$J_i = \frac{n_i}{\sum_{i=1}^{m} n_i} \sum_{i=1}^{m} J_i$$

(2) SV(Sales Value) method

$$b_i = \frac{p_i}{\sum_{i=1}^{m} p_i} \sum_{i=1}^{m} b_i$$

(3) Moriarity approach

$$T_i = \frac{Y_i}{\sum_{i=1}^{m} Y_i} \sum_{i=1}^{m} T_i$$

(4) Louderback approach

$$J_i = \frac{X_i - f_i}{X' - F'} \sum_{i=1}^{m} J_i \qquad \text{if } X_i > f_i$$

$$(\text{where } X' = \sum_{i=1}^{m} X_i \qquad F' = \sum_{i=1}^{m} f_i)$$

All approaches have some desirable characteristic and undesirable characteristics in decision making processes. Detailes of them are beyond the scope of this paper. Let's now use simple example to compare each joint cost allocation method.

(Example 2)

		f	p	X	n	Y
Joint cost 21	Product 1 (3 units)	14	19	16	5	16
	Product 2 (5 units)	3	29	27	26	24

Characteristic function(depends on the output vector α).

$$H_1(x_1,\ x_2) = x_1 + x_2 = R_1(x_1,\ x_2)$$

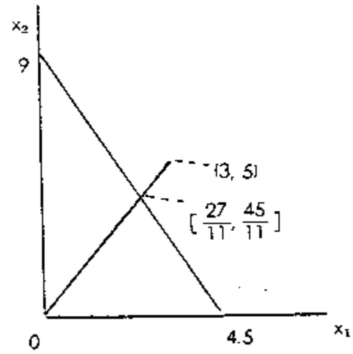
$$H_2(x_1,\ x_2) = 3x_1 + 2x_2 - 9 = R_2(x_1,\ x_2) - 9$$

The results are as follows.

	Product	NRV	SV	Moriarity	Louderback	A-S Mechanism
Joint	1	3.39	1.04	1.2	1.6	0.9
Cost	2	17.61	19.96	19.8	19.4	20.1
Total	1	17.39	15.04	15.2	15.6	14.9
Cost	2	20.61	22.96	22.8	22.4	23.1
Book	1	1.61	3.96	3.8	3.4	4.1
Profit	2	8.39	6.04	6.2	6.6	5.9

Note) 1. To get NRV, SV, Moriarity, Loderback cost allocation is straightforward.

2. By A-S Mechanism, see below.



From the figure $t_0=0$, $t_1=0.82$, $t_2=1$

$V(3, 5)=3\times3+2\times5-9=10$

$=$Total book profit

$$B_1=0.82\frac{\partial R_1}{\partial x_1}+0.18\frac{\partial R_2}{\partial x_1}$$

$$=0.82\times1+0.183\times3=1.36$$

$$B_2=0.82\frac{\partial R_1}{\partial x_2}+0.18\frac{\partial R_2}{\partial x_2}$$

$$=0.82\times1+0.18\times2=1.18$$

Threfore, for product 1, 4.1(1.36×3) of book profit is assigned and for product 2, 5.9(1.18×5) is assigned.

Let's call a joint cost allocation method that calculates book profit from selling prices "a gross method" and that calculates book profit from selling price less further processing cost "a net method". Then NRV and Louderback approaches are net method while SV and Moriarity approaches are gross method. Again both Louderback and Moriarity approaches are based on alternative costs, where NRV and SV approaches are not. From the interpretation of A-S profits, we can say that A-S profit mechanism are net method and also based on alternative costs. While Louderback and A-S profit mechanism are based on the same ground, A-S profit is calculated through the marginal profit of goods whereas Louderback is not.

5. Discussion

In this paper, we apply A-S price mechanism to the profit function for the allocation of joint cost through book profit. Meanwhile, we must know the profit function as well as the cost function. To get the exact form of thest two functions is not easy job in real situation.

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