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Bayesian Estimation for Reliability in a System Consisting of the Left Truncated Exponential Components

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ABSTRACT

In this paper, we propose the Bayes estimators of the reliability for a system consisting of the left-truncated exponential components under the truncated normal distribution as a conjugate prior distribution and squared-error loss function on the series, parallel and k-out-of-m : G system. And we compare the proposed Bayes estimators of the system reliability each other in terms of MSE performances and stabilities by the Monte Carlo simulation.

1. Introduction

We consider a system consistng of m independent and identical components each of which the lifetimes are distributed as the left truncated exponential distribution (LTED). It is well-known that the LTED is appropriate as a lifetime distribution model since the guaranteed lifetime which is a period with no initial failures is considered. The probability density function of the LTED with positivity constraint on the threshold parameter is given by

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$$f(y|\lambda, \theta) = \frac{1}{\lambda}\exp[-\frac{1}{\lambda}(y-\theta)], \quad y>\theta, \quad \lambda>0, \tag{1.1}$$

where $\theta>0$ for the density to be truncated.
The scale parameter λ is the reciprocal of the failure rate and the threshold parameter θ represents a guaranteed lifetime, respectively. This model will be referred to as LTE(λ, θ).

For the LTED, the component reliability which is the probability that performs until a given time t is given by

$$R_c(t) = P_r(Y>t)$$

$$= \exp\left(-\frac{t-\theta}{\lambda}\right), \quad t>\theta. \tag{1.2}$$

The Bayesian approaches to the estimation of the reliability for a left truncated exponential component were considered by many authors.

Varde(1969) compared the Bayes estimators under the two kinds of loss function and a conjugate prior distribution with the classical estimators of the reliability for the LTED. Sinha and Guttman(1976) proposed the Bayes estimators of the reliability under the squared-error loss and the general classes of the noninformative prior distributions, and derived the posterior bounds of the reliability for the LTED. Pierce(1973) considered the Bayesian reliability estimation for the LTED assuming the prior distribution is a special case of the noninformative prior distribution used by Sinha and Guttman(1976). In recent year, Trader(1985) discussed the Bayes estimators of the reliability under the squared-error loss and truncated normal distribution as a conjugate prior distribution for the LTED. Also, Ellis and Tummala(1986) compared the performances of the Bayes maximum likelihood (BML) estimator with the other estimators of the reliability for the Weibull lifetime distribution.

And the complete discussions of the Bayesian reliability estimator for the series, parallel and k-out-of-m system consisting of the one-parameter exponential components under the conjugate prior distribution of the reliability can be found in Martz and Waller(1982). Also, Basu and Mawaziny(1978) studied the minimum variance unbiased (MVU) estimator of the reliability in a k-out-of-m system consisting of the one-parameter exponential components, and compared the performances of the MVUE of reliability with the corresponding maximum likelihood (ML) estimator. In 1981, Chao(1981) proposed the approximate mean squared-errors (MSEs) of the estimators of the reliability for a k-out-of-m system consisting of the one-parameter exponential components, and compared with the corresponding ML estimator.

In the Bayesian viewpoint, Chao And Hwang(1983) proposed a simple approximation formula for the MSE of the Bayes estimators of the reliability under the noninformative prior distribution of the parameters in a k-out-of-m system.

The purpose of this thesis is to propose the Bayes estimators of the reliability for a system consisting of the left truncated exponential components under the truncated normal distribution as a conjugate prior distribution and squared-error loss function on some system configurations ; series, parallel and k-out-of-m, and to compare the proposed Bayes estimators of the system reliability each other in terms of the MSE performances and stabilities by the Monte Carlo simulation.

2. Preliminaries

Let $x_{(1)}<x_{(2)}<\cdots<x_{(r)}$ be the first r ordered observed lifetimes of n lifetimes from a system of m identical components each of which the lifetimes are distributed as LTE(λ, θ). The n_i replicates of the i-th components are life-tested and the life test is terminated at cummulative failures r of these $n=\sum_{i=1}^{m} n_i$, $r \leq n$.

Assuming that lifetimes are statistically independent, the likelihood function of λ and θ can be written as

$$L(\lambda, \theta|x_{(1)}, v_r)=\frac{n!}{(n-r)!}\lambda^{-r}\exp\left[-\frac{n(x_{(1)}-\theta)+v_r}{\lambda}\right], \tag{2.1}$$

where $v_r=\sum_{i=1}^{r}(x_{(i)}-x_{(1)})+(n-r)(x_{(r)}-x_{(1)})$, $\lambda>0$ and $0<\theta<x_{(1)}$.

To find a conjugate prior distribution for (λ, θ), especially, which is appropriate in the sence of the guaranteed life θ, we assume that Λ is distributed as inverted gamma distribution $IG(a, b)$ and $\Theta|\lambda$ is distributed as normal distribution $N\left(c, \frac{\lambda}{d}\right)$ which is truncated to the left at θ. Then the prior distributions of λ and $\Theta|\lambda$ are as follows ;

$$g_1(\lambda)=\frac{b^a}{\Gamma(a)}\lambda^{-(a+1)}\exp\left(-\frac{b}{\lambda}\right),$$

$$g_2(\theta|\lambda)=\frac{1}{\sqrt{2\pi}}\left(\frac{d}{\lambda}\right)^{\frac{1}{2}}\left[1-\Phi\left(-c\sqrt{\frac{d}{\lambda}}\right)\right]^{-1}\exp\left[-\frac{d}{2\lambda}(\theta-c)^2\right], \tag{2.2}$$

respectively, where $\theta, \lambda>0$, $a, b, d>0$, $-\infty<c<\infty$, and $\Phi(\cdot)$ is the standard normal cummulative distribution function (CDF).

From (2.2), we obtain the joint prior distribution of (λ, θ) as

$$g(\lambda, \theta)=g_1(\lambda)g_2(\theta|\lambda)$$

$$\propto d^{\frac{1}{2}}\lambda^{-(a+\frac{3}{2})}\exp\left\{-\frac{1}{\lambda}\left[\frac{d}{2}(\theta-c)^2+b\right]\right\}, \tag{2.3}$$

where θ, $\lambda>0$, a, b, $d>0$, respectively, and $-\infty<c<\infty$.

By means of the likelihood function in (2.1) and joint prior distribution in (2.3), we obtain the following lemma.

Lemma 2.1. The joint posterior distribution of (λ, θ) given $x_{(1)}$ and v_r is given by

$$h(\lambda, \theta|x_{(1)}, v_r)=\frac{\sqrt{2}b_1^{a_1}}{\Gamma(a_1+\frac{1}{2})B(a_1,\frac{1}{2})\{F_\beta(A_2|a_1,\frac{1}{2})-F_\beta(A_1|a_1,\frac{1}{2})\}}$$

$$\cdot d^{\frac{1}{2}}\lambda^{-(a_1+\frac{3}{2})}\exp\left\{-\frac{1}{\lambda}\left[\frac{d}{2}(\theta-c_1)^2+b_1\right]\right\}, \qquad (2.4)$$

where $\lambda>0$, $0<\theta<x_{(1)}<c_1$; $a_1=a+r$, $b_1=b+v_r+n(x_{(1)}-c)-\frac{n^2}{2d}$, $c_1=c+\frac{n}{d}$, $A_1=\left(1+\frac{d}{2b_1}c_1^2\right)^{-1}$, $A_2=\left[1+\frac{d}{2b_1}(x_{(1)}-c_1)^2\right]^{-1}$; $\Gamma(\cdot)$ and $B(\cdot)$ are gamma and beta functions, respectively, and $F_\beta(\cdot)$ is the beta CDF.

Proof. From (2.1) and (2.3), the posterior distribution of (λ, θ) becomes

$$h(\lambda, \theta \mid x_{(1)}, v_r)=\frac{L(\lambda, \theta|x_{(1)}, v_r)g(\lambda, \theta)}{\int_0^{x(1)}\int_0^\infty L(\lambda, \theta|x_{(1)}, v_r)g(\lambda, \theta)\,d\lambda d\theta}$$

$$=\frac{\left(\frac{d}{\lambda}\right)^{\frac{1}{2}}\lambda^{-(a+r+1)}}{\int_0^{x(1)}\int_0^\infty\left(\frac{d}{\lambda}\right)^{\frac{1}{2}}\lambda^{-(a+r+1)}}$$

$$\cdot\frac{\exp\left\{-\frac{1}{\lambda}\left[\frac{d(\theta-c)^2}{2}+n(x_{(1)}-\theta)+b+v_r\right\}\right.}{\exp\left\{-\frac{1}{\lambda}\left[\frac{d(\ -c)^2}{2}+n(x_{(1)}-\theta)+b+v_r\right]\right\}d\lambda d\theta}.$$

Letting $a_1=a+r$, $b_1=b+v_r+n(x_{(1)}-c)-\frac{n^2}{2d}$, respectively, and $c_1=c+\frac{n}{d}$, we obtain

$$h(\lambda, \theta|x_{(1)}, v_r)=\frac{\lambda^{-(a_1+\frac{3}{2})}\exp\left\{-\frac{1}{\lambda}\left[\frac{d}{2}(\theta-c_1)^2+b_1\right]\right\}}{\int_0^{x(1)}\int_0^\infty\lambda^{-(a_1+\frac{3}{2})}\exp\left\{-\frac{1}{\lambda}\left[\frac{d}{2}(\theta-c_1)^2+b_1\right]\right\}d\lambda d\theta}. \qquad (2.5)$$

The denominator of (2.5) reduces to

$$\int_0^{x(1)}\int_0^{\infty}\lambda^{-(a_1+\frac{3}{2})}\exp\left\{-\frac{1}{\lambda}\left[\frac{d}{2}(\theta-c_1)^2+b_1\right]\right\}d\lambda d\theta$$

$$=\Gamma\left(a_1+\frac{1}{2}\right)\int_0^{x(1)}\frac{1}{\left[\frac{d}{2}(\theta-c_1)^2+b_1\right]^{a_1+\frac{1}{2}}}\,d\theta$$

$$=\frac{\Gamma\left(a_1+\frac{1}{2}\right)}{b_1^{a_1+\frac{1}{2}}}\int_0^{x(1)}\left[\frac{1}{1+\frac{d}{2b_1}(\theta-c_1)^2}\right]^{a_1+\frac{1}{2}}d\theta,$$

which, using the transformation $z^{-1}=1+\frac{d}{2b_1}(\theta-c_1)^2$, may be written as

$$\int_0^{x(1)}\int_0^{\infty}\lambda^{-(a_1+\frac{3}{2})}\exp\left\{-\frac{1}{\lambda}\left[\frac{d}{2}(\theta-c_1)^2+b_1\right]\right\}d\lambda d\theta$$

$$=\frac{\Gamma\left(a_1+\frac{1}{2}\right)}{b_1^{a_1}\sqrt{2d}}\int_{[1+\frac{d}{2b_1}c_1^2]^{-1}}^{[1+\frac{d}{2b_1}(x_{(1)}-c_1)^2]^{-1}}z^{a_1-1}(1-z)^{-\frac{1}{2}}dz$$

$$=\frac{\Gamma\left(a_1+\frac{1}{2}\right)}{b_1^{a_1}\sqrt{2d}}B\left(a_1,\ \frac{1}{2}\right)\left\{F_\beta\left(A_2|a_1,\ \frac{1}{2}\right)-F_\beta\left(A_1|a_1,\ \frac{1}{2}\right)\right\},$$

which proves the lemma.

Remark 2.2. From the joint prior distribution of $(\lambda,\ \theta)$ in (2.3) and the joint posterior distribution of $(\lambda,\ \theta)$ in (2.4), we find that these are members of same family of distribution. Hence, the joint prior distribution in (2.3) is a natural conjugate prior distribution of $(\lambda,\ \theta)$.

3. Series System

The reliability for a system of m components connected in series system which is the failure of any component causes the system to fail can be written as

$$R_S(t)=P_r(X>t)$$

$$=\prod_{j=1}^{m}P_r(Y_j>t)\ =\prod_{j=1}^{m}R_c(t) \qquad (3.1)$$

where X is the lifetime of the series system, Y_j is the lifetime of the j-th component, respectively, and $R_C(t)$ is the reliability of the component.

From (3.1), the reliability for the series system of m identical components each of which lifetimes are distributed as LTE(λ, θ) is given by

$$R_s(t) = \exp\left[-\frac{m(t-\theta)}{\lambda}\right], \quad t > \theta.$$

Therefore, we obtain the following theorems ;

Theorem 3.1. Under the squared-error loss function and the conjugate prior distribution of (λ, θ) in (2.3), Bayes estimator of the series system reliability is given by

$$R_S^{BE}(t) = \frac{b_1^{a_1}\left\{F_\beta\left(A_4(t)\middle|\ a_1, \frac{1}{2}\right) - F_\beta\left(A_3(t)\middle|\ a_1, \frac{1}{2}\right)\right\}}{b_2(t)^{a_1}\left\{F_\beta\left(A_2\middle|\ a_1, \frac{1}{2}\right) - F_\beta\left(A_1\middle|\ a_1, \frac{1}{2}\right)\right\}}, \tag{3.2}$$

where $a_1 = a + r$, $b_1 = b + v_r + n(x_{(1)} - c) - \frac{n^2}{2d}$, $c_1 = c + \frac{n}{d}$, $x_{(1)} \le c_1$, $b_2(t) = b_1 + mt - mc_1 - \frac{m^2}{2d}$, $c_2 = c_1 + \frac{m}{d}$, $A_1 = \left(1 + \frac{d}{2b_1}c_1^2\right)^{-1}$, $A_2 = \left[1 + \frac{d}{2b_1}(x_{(1)} - c_1)^2\right]^{-1}$, $A_3(t) = \left(1 + \frac{d}{2b_2(t)}c_2^2\right)^{-1}$, $A_4(t) = \left[1 + \frac{d}{2b_2(t)}(x_{(1)} - c_2)^2\right]^{-1}$, respectively, and $F_\beta(\cdot)$ is the beta CDF.

Proof. Since the Bayes estimator of the series system reliability is the mean of the posterior distribution of (λ, θ) given $x_{(1)}$ and v_r in (2.4), we can find the Bayes estimator of the series system reliability as follows ;

$$R_S^{BE}(t) = \int_0^{x(1)}\int_0^\infty R_s(t)\,h(\lambda, \theta|x_{(1)}, v_r)\,d\lambda d\theta$$

$$= \frac{b_1^{a_1}\sqrt{2d}}{\Gamma(a_1 + \frac{1}{2})B(a_1, \frac{1}{2})\{F_\beta(A_2|a_1, \frac{1}{2}) - F_\beta(A_1|a_1, \frac{1}{2})\}}$$

$$\cdot \int_0^{x(1)}\int_0^\infty \lambda^{-(a_1+\frac{3}{2})}\exp\left\{-\frac{1}{\lambda}\left[\frac{d}{2}(\theta - c_1)^2 - m\theta + b_1 + mt\right]\right\}d\lambda d\theta.$$

Letting $b_2(t) = b_1 + mt - mc_1 - \frac{m^2}{2d}$, $c_2 = c_1 + \frac{m}{d}$, respectively, and making the transformation $z^{-1} = 1 + \frac{d}{2b_2(t)}(\theta - c_2)^2$, the last integral may be written as

$$\int_0^{x(1)}\int_0^{\infty}\lambda^{-(a_1+\frac{3}{2})}\exp\left\{-\frac{1}{\lambda}\left[\frac{d}{2}(\theta-c_1)^2-m\theta+b_1+mt\right]\right\}d\lambda d\theta$$

$$=\int_0^{x(1)}\int_0^{\infty}\lambda^{-(a_1+\frac{3}{2})}\exp\left\{-\frac{1}{\lambda}\left[\frac{d}{2}(\theta-c_2)^2+b_2(t)\right]\right\}d\lambda d\theta$$

$$=\frac{\Gamma\left(a_1+\frac{1}{2}\right)}{b_2(t)^{a_1}}\int_0^{x(1)}\left[\frac{1}{1+\frac{d}{2b_2(t)}(\theta-c_2)^2}\right]^{a_1+\frac{1}{2}}d\theta$$

$$=\frac{\Gamma\left(a_1+\frac{1}{2}\right)}{b_2(t)^{a_1}\sqrt{2d}}\int_{[1+\frac{d}{2b_2(t)}c_2^2]^{-1}}^{[1+\frac{d}{2b_2(t)}(x_{(1)}-c_2)^2]^{-1}} z^{a_1-1}(1-z)^{-\frac{1}{2}}\,dz$$

$$=\frac{\Gamma\left(a_1+\frac{1}{2}\right)}{b_2(t)^{a_1}\sqrt{2d}}B\left(a_1,\ \frac{1}{2}\right)\left\{F_\beta\left(A_4|a_1,\ \frac{1}{2}\right)-F_\beta\left(A_3(t)\,|a_1,\ \frac{1}{2}\right)\right\}.$$

Hence we have the theorem 3. 1.

Theorem 3.2. Under the conjugate prior distribution of $(\lambda,\ \theta)$ in (2. 3), the Bayes ML estimator of the series system reliability is given by

$$R_S^{BMLE}(t)=\exp\left[-\frac{m}{b_1}(t-c_1)\left(a_1+\frac{3}{2}\right)\right], \tag{3. 3}$$

where $a_1=a+r$, $b_1=b+v_r+n(x_{(1)}-c)-\frac{n^2}{2d}$, $c_1=c+\frac{n}{d}$, respectively, and $x_{(1)}\leq c_1$.

Proof. Maximizing the joint posterior distribution of $(\lambda,\ \theta)$ in (2. 4) with respect to λ and θ, we can obtain the equation (3. 3).

Remark 3.3. If $a=-\frac{3}{2}$, $b=0$, $c=x_{(1)}$, respectively, and $d=\infty$ in the Bayes ML estimator of the series system reliability in (3. 3), then $R_S^{BMLE}(t)$ is identical with the classical estimator of the series system reliability in Sinha (1985).

4. Parallel System

The reliability for a system of m components connected in parallel system which is the failure of all components causes the system to fail can be written as

$$R_p(t) = 1 - P_r(X \le t)$$

$$= 1 - \prod_{j=1}^{m} [1 - P_r(Y_j > t)]$$

$$= 1 - \prod_{j=1}^{m} [1 - R_c(t)] \qquad (4.1)$$

where X is the lifetime of the parallel system, Y_j is the lifetime of the j-th component, respectively, and $R_c(t)$ is the reliability of the component.

From (4.1), the reliability for a parallel system of m identical components each of which lifetimes are distributed as LTE(λ, θ) is given by

$$R_p(t) = 1 - \sum_{j=0}^{m} (-1)^j \binom{m}{j} \exp\left[-\frac{j(t-\theta)}{\lambda}\right], \quad t > \theta.$$

Hence, we have the following theorems :

Theorem 4.1. Under the squared-error loss function and the conjugate prior distribution of (λ, θ) in (2.3), Bayes estimator of the parallel system reliability is given by

$$R_p^{BE}(t) = 1 - \sum_{j=0}^{m} (-1)^j \frac{b_1^{a_1}\left\{F_\beta\left(A_{6j}(t) \mid a_1, \frac{1}{2}\right) - F_\beta\left(A_{5j}(t) \mid a_1, \frac{1}{2}\right)\right\}}{b_{3j}(t)^{a_1}\left\{F_\beta\left(A_2 \mid a_1, \frac{1}{2}\right) - F_\beta\left(A_1 \mid a_1, \frac{1}{2}\right)\right\}}, \qquad (4.2)$$

where $a_1 = a + r$, $b_1 = b + v_r + n(x_{(1)} - c) - \frac{n^2}{2d}$, $c_1 = c + \frac{n}{d}$, $x_{(1)} \le c_1$, $b_{3j}(t) = b_1 + jt - jc_{3j} - \frac{j^2}{2d}$, $c_{3j} = c_1 + \frac{j}{d}$, $A_1 = \left(1 + \frac{d}{2b_1} c_1^2\right)^{-1}$, $A_2 = \left[1 + \frac{d}{2b_1}(x_{(1)} - c_1)^2\right]^{-1}$, $A_{6j}(t) = \left(1 + \frac{d}{2b_{3j}(t)} c_{3j}^2\right)^{-1}$, $A_{5j}(t) = \left[1 + \frac{d}{2b_{3j}(t)}(x_{(1)} - c_{3j})^2\right]^{-1}$, respectively, and $F_\beta(\cdot)$ is the beta CDF.

Proof. Since the Bayes estimator of the parallel system reliability is the mean of the posterior distribution of (λ, θ) given $x_{(1)}$ and v_r in (2.4), we can find the Bayes estimator of the parallel system reliability as follows ;

$$R_p^{BE}(t) = \int_0^{x(1)} \int_0^{\infty} R_p(t)\, h(\lambda, \theta \mid x_{(1)}, v_r)\, d\lambda d\theta$$

$$= 1 - \sum_{j=0}^{m} (-1)^j \frac{b_1^{a_1}\sqrt{2d}}{\Gamma(a_1 + \frac{1}{2}) B(a_1, \frac{1}{2}) \{F_\beta(A_2 \mid a_1, \frac{1}{2}) - F_\beta(A_1 \mid a_1, \frac{1}{2})\}}$$

$$\cdot \int_0^{x(1)} \int_0^{\infty} \lambda^{-(a_1 + \frac{3}{2})} \exp\left\{-\frac{1}{\lambda}\left[\frac{d}{2}(\theta - c_1)^2 - j\theta + b_1 + jt\right]\right\} d\lambda d\theta.$$

Letting $b_{3j}(t)=b_1+jt-jc_1-\frac{j^2}{2d}$, $c_{3j}=c_1+\frac{j}{d}$, respectively, and making the transformation $z^{-1}=1+\frac{d}{2b_{3j}(t)}(\theta-c_{3j})^2$, the last integral may be written as

$$\int_0^{x(1)}\int_0^{\infty}\lambda^{-(a1+\frac{3}{2})}\exp\left\{-\frac{1}{\lambda}\left[\frac{d}{2}(\theta-c_1)^2-j\theta+b_1+jt\right]\right\}d\lambda d\theta$$

$$=\int_0^{x(1)}\int_0^{\infty}\lambda^{-(a1+\frac{3}{2})}\exp\left\{-\frac{1}{\lambda}\left[\frac{d}{2}(\theta-c_{3j})^2+b_{3j}(t)\right]\right\}d\lambda d\theta$$

$$=\frac{\Gamma\left(a_1+\frac{1}{2}\right)}{b_{3j}(t)^{a_1}}\int_0^{x(1)}\left[\frac{1}{1+\frac{d}{2b_{3j}(t)}(\theta-c_{3j})^2}\right]^{a1+\frac{1}{2}}d\theta$$

$$=\frac{\Gamma\left(a_1+\frac{1}{2}\right)}{b_{3j}(t)^{a_1}\sqrt{2d}}\int_{[1+\frac{d}{2b_{3j}(t)}c_{3j}^2]^{-1}}^{[1+\frac{d}{2b_{3j}(t)}(x_{(1)}-c_{3j})^2]^{-1}}z^{a_1-1}(1-z)^{-\frac{1}{2}}dz$$

$$=\frac{\Gamma\left(a_1+\frac{1}{2}\right)}{b_{3j}(t)^{a_1}\sqrt{2d}}B\left(a_1,\ \frac{1}{2}\right)\left\{F_\beta\left(A_{6j}(t)|a_1,\ \frac{1}{2}\right)-F_\beta\left(A_{5j}(t)|a_1,\ \frac{1}{2}\right)\right\}.$$

which proves the theorem.

Theorem 4.2. Under the conjugate prior distribution of $(\lambda,\ \theta)$ in (2.3), the Bayes ML estimator of the parallel system reliability is given by

$$R_p^{BMLE}(t)=1-\sum_{j=0}^{m}(-1)^j\exp\left[-\frac{j(t-c_1)(a_1+\frac{3}{2})}{b_1}\right], \tag{4.3}$$

where $a_1=a+r$, $b_1=b+v_r+n(x_{(1)}-c)-\frac{n^2}{2d}$, $c_1=c+\frac{n}{d}$, respectively, and $x_{(1)}\leq c_1$.

Proof. Maximizing the joint posterior distribution of $(\lambda,\ \theta)$ in (2.4) with respect to λ and θ, we can obtain the equation (4.3).

Remark 4.3. If $a=-\frac{3}{2}$, $b=0$, $c=x_{(1)}$, respectively, and $d=\infty$ in the Bayes ML estimator of the parallel system reliability in (4.3), then $R_p^{BMLE}(t)$ is identical with the classical estimator of the parallel system reliability in Sinha (1985).

5. k-out-of-m : G system

We consider a k-out-of-m : G system with m components such that the system operates if and only if at least k of these m components operate. The k-out-of-m : G system is more general than series or parallel system. The reliability for the k-out-of-m : G system can be written as

$$R_{k,m}(t)=\sum_{j=k}^{m}\binom{m}{j}R_c(t)^j[1-R_c(t)]^{m-j}, \tag{5.1}$$

where $R_c(t)$ is the reliability of the component.

From (5.1), the reliability for the k-out-of-m : G system of m identical components each of which lifetimes are distributed as LTE$(\lambda,\ \theta)$ is given by

$$R_{k,m}(t)=\sum_{j=k}^{m}\binom{m}{j}\left[\exp\left(-\frac{t-\theta}{\lambda}\right)\right]^j\left[1-\exp\left(-\frac{t-\theta}{\lambda}\right)\right]^{m-j}$$

$$=\sum_{j=k}^{m}\sum_{q=0}^{m-j}(-1)^q\binom{m}{j}\binom{m-j}{q}\exp\left[-\frac{(j+q)(t-\theta)}{\lambda}\right],\quad t>\theta.$$

Thus, we get the following theorems :

Theorem 5.1. Under the squared-error loss function and the conjugate prior distribution of $(\lambda,\ \theta)$ in (2.3), Bayes estimator of the k-out-of-m : G system reliability is given by

$$R_{k,m}^{BE}(t)=\sum_{j=k}^{m}\sum_{q=0}^{m-j}\binom{m}{j}\binom{m-j}{q}(-1)^q$$

$$\cdot\frac{b_1^{a_1}\left\{F_\beta\left(A_{8jq}(t)|a_1,\ \frac{1}{2}\right)-F_\beta\left(A_{7jq}(t)|a_1,\ \frac{1}{2}\right)\right\}}{b_{4jq}(t)^{a_1}\left\{F_\beta\left(A_2|a_1,\ \frac{1}{2}\right)-F_\beta\left(A_1|a_1,\ \frac{1}{2}\right)\right\}}, \tag{5.2}$$

where $a_1=a+r$, $b_1=b+v_r+n(x_{(1)}-c)-\frac{n^2}{2d}$, $c_1=c+\frac{n}{d}$, $x_{(1)}\le c_1$, $b_{4jq}(t)=b_1+(j+q)t-(j+q)c_1-\frac{(j+q)^2}{2d}$, $c_{4jq}=c_1+\frac{(j+q)}{d}$, $A_1=\left(1+\frac{d}{2b_1}c_1^2\right)^{-1}$, $A_2=\left[1+\frac{d}{2b_1}(x_{(1)}-c_1)^2\right]^{-1}$, $A_{7jq}(t)=\left(1+\frac{d}{2b_{4jq}(t)}c_{4jq}^2\right)^{-1}$, $A_{8jq}(t)=\left[1+\frac{d}{2b_{4jq}(t)}(x_{(1)}-c_{4jq})^2\right]^{-1}$, respectively, and $F_\beta(\cdot)$ is the beta CDF.

Proof. Since the Bayes estimator of the k-out-of-m : G system reliability is the mean of the posterior distribution of (λ, θ) given $x_{(1)}$ and v_r in (2.4), we can get the Bayes estimator of the k-out-of-m : G system reliability as follows ;

$$R^{BE}_{k,m}(t)=\int_0^{x(1)}\int_0^{\infty}R_{k,m}(t)\,h(\lambda,\ \theta|x_{(1)},\ v_r)\,d\lambda d\theta$$

$$=\sum_{j=k}^{m}\sum_{q=0}^{m-j}(-1)^q\binom{m}{j}\binom{m-j}{q}$$

$$\cdot\frac{b_1^{a_1}\sqrt{2d}}{\Gamma\left(a_1+\frac{1}{2}\right)B\left(a_1,\ \frac{1}{2}\right)\left\{F_\beta\left(A_2|\ a_1,\ \frac{1}{2}\right)-F_\beta\left(A_1|\ a_1,\ \frac{1}{2}\right)\right\}}$$

$$\cdot\int_0^{x(1)}\int_0^{\infty}\lambda^{-(a_1+\frac{3}{2})}\exp\left\{-\frac{1}{\lambda}\left[\frac{d}{2}(\theta-c_1)^2-(j+q)\theta+b_1+(j+q)t\right]\right\}d\lambda d\theta.$$

Letting $b_{4jq}(t)=b_1+(j+q)t-(j+q)c_1-\frac{(j+q)^2}{2d}$, $c_{4jq}=c_1+\frac{(j+q)}{d}$, respectively, and making the transformation $z^{-1}=1+\frac{d}{2b_{4jq}(t)}(\theta-c_{4jq})^2$, the last integral may be written as

$$\int_0^{x(1)}\int_0^{\infty}\lambda^{-(a_1+\frac{3}{2})}\exp\left\{-\frac{1}{\lambda}\left[\frac{d}{2}(\theta-c_1)^2-(j+q)\theta+b_1+(j+q)t\right]\right\}d\lambda d\theta$$

$$=\int_0^{x(1)}\int_0^{\infty}\lambda^{-(a_1+\frac{3}{2})}\ \exp\left\{-\frac{1}{\lambda}\left[\frac{d}{2}(\theta-c_{4jq})^2+b_{4jq}(t)\right]\right\}d\lambda d\theta$$

$$=\frac{\Gamma\left(a_1+\frac{1}{2}\right)}{b_{4jq}(t)^{a_1}}\int_0^{x(1)}\left[\frac{1}{1+\frac{d}{2b_{4jq}(t)}(\theta-c_{4jq})^2}\right]^{a_1+\frac{1}{2}}d\theta$$

$$=\frac{\Gamma\left(a_1+\frac{1}{2}\right)}{b_{4jq}(t)^{a_1}\sqrt{2d}}\int_{[1+\frac{d}{2b_{4jq}(t)}c_{4jq}^2]^{-1}}^{[1+\frac{d}{2b_{4jq}(t)}(x_{(1)}-c_{4jq})^2]^{-1}}z^{a_1-1}(1-z)^{-\frac{1}{2}}dz$$

$$=\frac{\Gamma\left(a_1+\frac{1}{2}\right)}{b_{4jq}(t)^{a_1}\sqrt{2d}}B\left(a_1,\ \frac{1}{2}\right)\left\{F_\beta\left(A_{8jq}(t)|a_1,\ \frac{1}{2}\right)-F_\beta\left(A_{7jq}(t)|a_1,\ \frac{1}{2}\right)\right\}.$$

Thus, we have the theorem 5.1.

Remark 5.2. If we take $k=1$ and $k=m$ in the Bayes estimator of the k-out-of-m : G system reliability in (5.2), then $R^{BE}_{1,m}(t)$ is identical with $R^{BE}_{p}(t)$ in (4.2) and $R^{BE}_{m,m}(t)$ is identical with $R^{BE}_{s}(t)$ in (3.2), respectively.

Theorem 5.3. Under the conjugate prior distribution of (λ, θ) in (2.3), the Bayes ML estimator of the k-out-of-m : G system reliability is given by

$$R^{BMLE}_{k,m}(t) = \sum_{j=k}^{m} \sum_{q=0}^{m-j} (-1)^q \binom{m}{j}\binom{m-j}{q} \exp\left[-\frac{(j+q)(t-c_1)(a_1+\frac{3}{2})}{b_1}\right], \qquad (5.3)$$

where $a_1 = a + r$, $b_1 = b + v_r + n(x_{(1)} - c) - \frac{n^2}{2d}$, $c_1 = c + \frac{n}{d}$, respectively, and $x_{(1)} \le c_1$.

Proof. Maximizing the joint posterior distribution of (λ, θ) in (2.4) with respect to λ and θ, we can obtain the equation (5.3).

Remark 5.4. If we take $k=1$ and $k=m$ in the Bayes ML estimator of the k-out-of-m : G system reliability in (5.3), then $R^{BMLE}_{1,m}(t)$ is identical with $R^{BMLE}_{p}(t)$ in (4.3) and $R^{BMLE}_{m,m}(t)$ is identical with $R^{BMLE}_{s}(t)$ in (3.3), respectively.

Remark 5.5. If $a = -\frac{3}{2}$, $b=0$, $c = x_{(1)}$, respectively, and $d=\infty$ in the Bayes ML estimatoi of the k-out-of-m : G system reliability in (5.3), then $R^{BMLE}_{k,m}(t)$ is identical with the classical estimator of the k-out-of-m : G system reliability in Chao and Hwang (1983).

6. Monte Carlo Comparisions

In this section, we compare the proposed Bayes and Bayes ML estimators of reliability for a system consisting of the left-truncated exponential components on series, parallel, and k-out-of-m : G system each other in terms of the MSE performances and stabilities by the Monte Carlo simulation. And the efficiencies of the proposed Bayes estimators measured in terms of the ratio of the MSEs for these system reliability.

We generate random samples from the LTE(3, 2) and we evaluate the Bayes estimators of the system reliability on these samples. We evaluate the estimates of biases, MSEs and efficiencies of the Bayes estimators with respect to classical ML estimators of system reliability based on 500 repetitions.

Simulations are performed on HP-9845 at the National Fisheries University of Pusan and the results of simulations appear in tables.

From the tables, the results can be summarized as follows :

(1) As the results in Basu and Mawaziny (1978), in general, an estimator is not uniformly

better than the other for k-out-of-3 systems ($k=1, 2, 3$).

(2) For the $R_{k,m}^{BMLE}$, the smaller d is for all k, m, a, b, c and t, the smaller MSE is.

(3) For the $R_{k,m}^{BMLE}$, the larger a and b are for all k, m, c, d and t, the smaller MSE is.

(4) For all c and t, a, b and d are larger, MSEs of $R_{k,m}^{BE}$ and $R_{k,m}^{BMLE}$ are smaller in series system, whereas MSE of $R_{k,m}^{BE}$ is larger and MSE of $R_{k,m}^{BMLE}$ is smaller in parallel system.

Table 1. Comparisons of Bayes and Bayes ML estimators of the system reliability for 1-out-of-3 : G system.

(1) $t=3.0$

(a, b, c, d)	(3, 10, 2, 15)		(3, 10, 2, 30)		(5, 20, 2, 15)		(5, 20, 2, 30)		
Estimators	$R_{1,3}^{BE}(t)$	$R_{1,3}^{BMLE}(t)$	$R_{1,3}^{BE}(t)$	$R_{1,3}^{BMLE}(t)$	$R_{1,3}^{BE}(t)$	$R_{1,3}^{BMLE}(t)$	$R_{1,3}^{BE}(t)$	$R_{1,3}^{BMLE}(t)$	$R_{1,3}^{MLE}(t)$
Bias	0.0458	0.0293	0.0590	0.0565	0.0196	−0.0047	0.0252	0.0074	0.1106
MSE*	0.2202	0.1043	0.3660	0.3558	0.0039	0.0003	0.0065	0.0007	6.4615
efficiency	0.0341	0.0161	0.0566	0.0551	0.0068	0.0005	0.0100	0.0012	1.0000

(* entry of MSE in table has been multiplied by 10^{-2})

(2) $t=5.0$

(a, b, c, d)	(3, 10, 2, 15)		(3, 10, 2, 30)		(5, 20, 2, 15)		(5, 20, 2, 30)		
Estimators	$R_{1,3}^{BE}(t)$	$R_{1,3}^{BMLE}(t)$	$R_{1,3}^{BE}(t)$	$R_{1,3}^{BMLE}(t)$	$R_{1,3}^{BE}(t)$	$R_{1,3}^{BMLE}(t)$	$R_{1,3}^{BE}(t)$	$R_{1,3}^{BMLE}(t)$	$R_{1,3}^{MLE}(t)$
Bias	0.1460	0.2951	0.1583	0.3099	−0.0013	0.0729	0.0147	0.0945	0.2759
MSE*	2.9573	10.1321	3.3150	10.9624	0.1717	0.8350	0.2055	1.2094	17.9932
efficiency	0.1644	0.5631	0.1842	0.6093	0.0095	0.0464	0.0114	0.0672	1.0000

(* entry of MSE in table has been multiplied by 10^{-2})

(3) $t=7.0$

(a, b, c, d)	(3, 10, 2, 15)		(3, 10, 2, 30)		(5, 20, 2, 15)		(5, 20, 2, 30)		
Estimators	$R_{1,3}^{BE}(t)$	$R_{1,3}^{BMLE}(t)$	$R_{1,3}^{BE}(t)$	$R_{1,3}^{BMLE}(t)$	$R_{1,3}^{BE}(t)$	$R_{1,3}^{BMLE}(t)$	$R_{1,3}^{BE}(t)$	$R_{1,3}^{BMLE}(t)$	$R_{1,3}^{MLE}(t)$
Bias	0.1308	0.3037	0.1320	0.3044	−0.0236	0.1164	−0.0141	0.1260	0.2193
MSE*	2.8169	10.2415	2.7873	10.2513	0.4757	1.9154	0.4318	2.1264	11.2634
efficiency	0.2501	0.9093	0.2475	0.9101	0.0422	0.1701	0.0383	0.1888	1.0000

(* entry of MSE in table has been multiplied by 10^{-2})

Table 2. Comparisons of Bayes and Bayes ML estimators of the system reliability for 2-out-of-3 : G system.

(1) $t=3.0$

(a, b, c, d)	(3, 10, 2, 15)		(3, 10, 2, 30)		(5, 20, 2, 15)		(5, 20, 2, 30)		
Estimators	$R_{2,3}^{BE}(t)$	$R_{2,3}^{BMLE}(t)$	$R_{2,3}^{BE}(t)$	$R_{2,3}^{BMLE}(t)$	$R_{2,3}^{BE}(t)$	$R_{2,3}^{BMLE}(t)$	$R_{2,3}^{BE}(t)$	$R_{2,3}^{BMLE}(t)$	$R_{2,3}^{MLE}(t)$
Bias	0.0308	0.0998	0.0814	0.1781	−0.0502	−0.0312	−0.0168	0.0271	0.1176
MSE*	0.2641	1.3603	0.9124	3.6369	0.2766	0.1431	0.0716	0.1460	9.9469
efficiency	0.0265	0.1368	0.0917	0.3655	0.0278	0.0144	0.0071	0.0147	1.0000

(* entry of MSE in table has been multiplied by 10^{-2})

(2) $t=5.0$

(a, b, c, d)	(3, 10, 2, 15)		(3, 10, 2, 30)		(5, 20, 2, 15)		(5, 20, 2, 30)		
Estimators	$R_{2,3}^{BE}(t)$	$R_{2,3}^{BMLE}(t)$	$R_{2,3}^{BE}(t)$	$R_{2,3}^{BMLE}(t)$	$R_{2,3}^{BE}(t)$	$R_{2,3}^{BMLE}(t)$	$R_{2,3}^{BE}(t)$	$R_{2,3}^{BMLE}(t)$	$R_{2,3}^{MLE}(t)$
Bias	0.0606	0.2039	0.0737	0.2115	−0.0632	0.0676	−0.0404	0.0867	0.1020
MSE*	1.1291	4.7199	1.2406	4.9655	0.7304	0.8243	0.4827	1.0899	6.6530
efficiency	0.1697	0.7094	0.1865	0.7464	0.1098	0.1239	0.0726	0.1638	1.0000

(* entry of MSE in table has been multiplied by 10^{-2})

(3) $t=7.0$

(a, b, c, d)	(3, 10, 2, 15)		(3, 10, 2, 30)		(5, 20, 2, 15)		(5, 20, 2, 30)		
Estimators	$R_{2,3}^{BE}(t)$	$R_{2,3}^{BMLE}(t)$	$R_{2,3}^{BE}(t)$	$R_{2,3}^{BMLE}(t)$	$R_{2,3}^{BE}(t)$	$R_{2,3}^{BMLE}(t)$	$R_{2,3}^{BE}(t)$	$R_{2,3}^{BMLE}(t)$	$R_{2,3}^{MLE}(t)$
Bias	0.0092	0.0790	0.0108	0.0794	−0.0518	0.0410	−0.0451	0.0442	0.0306
MSE*	0.2728	0.6812	0.2549	0.6824	0.4444	0.2554	0.3654	0.2729	1.2879
efficiency	0.2119	0.5289	0.1979	0.5298	0.3451	0.1983	0.2837	0.2119	1.0000

(* entry of MSE in table has been multiplied by 10^{-2})

Table 3. Comparisons of Bayes and Bayes *ML* estimators of the system reliability for 3-out-of-3 : *G* system.

(1) $t=3.0$

(a, b, c, d)	(3, 10, 2, 15)		(3, 10, 2, 30)		(5, 20, 2, 15)		(5, 20, 2, 30)		
Estimators	$R^{BE}_{3,3}(t)$	$R^{BMLE}_{3,3}(t)$	$R^{BE}_{3,3}(t)$	$R^{BMLE}_{3,3}(t)$	$R^{BE}_{3,3}(t)$	$R^{BMLE}_{3,3}(t)$	$R^{BE}_{3,3}(t)$	$R^{BMLE}_{3,3}(t)$	$R^{MLE}_{3,3}(t)$
Bias	−0.0515	0.1157	0.0449	0.1768	−0.1690	−0.0349	−0.0669	0.0396	0.0283
MSE*	0.4284	1.5110	0.3486	3.2614	2.8953	0.1746	0.4930	0.2080	6.6388
efficiency	0.0645	0.2276	0.0525	0.4913	0.4361	0.0263	0.0743	0.0313	1.0000

(* entry of MSE in table has been multiplied by 10^{-2})

(2) $t=5.0$

(a, b, c, d)	(3, 10, 2, 15)		(3, 10, 2, 30)		(5, 20, 2, 15)		(5, 20, 2, 30)		
Estimators	$R^{BE}_{3,3}(t)$	$R^{BMLE}_{3,3}(t)$	$R^{BE}_{3,3}(t)$	$R^{BMLE}_{3,3}(t)$	$R^{BE}_{3,3}(t)$	$R^{BMLE}_{3,3}(t)$	$R^{BE}_{3,3}(t)$	$R^{BMLE}_{3,3}(t)$	$R^{MLE}_{3,3}(t)$
Bias	−0.0036	0.0405	0.0019	0.0416	−0.0419	0.0169	−0.0307	0.0213	−0.0024
MSE*	0.0838	0.1776	0.0671	0.1839	0.2306	0.0506	0.1395	0.0630	1.1715
efficiency	0.0716	0.1516	0.0573	0.1570	0.1968	0.0432	0.1191	0.0538	1.0000

(* entry of MSE in table has been multiplied by 10^{-2})

(3) $t=7.0$

(a, b, c, d)	(3, 10, 2, 15)		(3, 10, 2, 30)		(5, 20, 2, 15)		(5, 20, 2, 30)		
Estimators	$R^{BE}_{3,3}(t)$	$R^{BMLE}_{3,3}(t)$	$R^{BE}_{3,3}(t)$	$R^{BMLE}_{3,3}(t)$	$R^{BE}_{3,3}(t)$	$R^{BMLE}_{3,3}(t)$	$R^{BE}_{3,3}(t)$	$R^{BMLE}_{3,3}(t)$	$R^{MLE}_{3,3}(t)$
Bias	−0.0058	0.0061	−0.0052	0.0061	−0.0155	0.0037	−0.0137	−0.0038	−0.0021
MSE*	0.0166	0.0041	0.0143	0.0041	0.0343	0.0025	0.0275	0.0025	0.0609
efficiency	0.2718	0.0674	0.2339	0.0672	0.5627	0.0405	0.4514	0.0409	1.0000

(* entry of MSE in table has been multiplied by 10^{-2})

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