Journal of the KSQC
Vol. 17, No. 2, 1989

On a Nonparametric Test for Parallelism against Ordered Alternatives+

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ABSTRACT

A nonparametric test for testing the parallelism of regression lines against ordered alternatives is proposed. The proposed test statistic is based on a linear combination of robust slope estimators. It is a modified version of the Adichie's test statistics based on scores. A small-sample Monte Carlo study shows that the proposed test is compatible with the Adichie's test.

1. Introduction

We are interested in the problem of testing the parallelism of several regression lines against ordered alternatives. The problem of ordered alternatives in slope parameters could arise in practice. For example, as stated in Adichie(1976), a biologist may be interested in knowing whether the rate of dependence of infection on exposure is the same for groups of rats of increasing ages.

+ This research was supported by the Ministry of Education, through the Research Institute for Basic Sciences, SNU 1988-1989.
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Consider the linear regression model

$$Y_{ij} = \alpha_i + \beta_i x_{ij} + \varepsilon_{ij}, \quad j=1, \cdots, n_i \ ; \ i=1, \cdots, k, \tag{1.1}$$

where the α_i's and β_i's are unknown parameters, the x_{ij}'s are known regression constants, and ε_{ij}'s are independent and identically distributed random variables with a continuous cumulative distribution function(cdf) F. Here, the α_i's are nuisance parameters and the β_i's are the slope parameters of interest.

In this paper, we are concerned with the problem of testing for the parallelism of k regression lines against ordered alternatives. That is, we want to test

$$H_0 : \beta_1 = \cdots = \beta_k = \beta \text{(unknown)} \ ; \ \alpha_i \text{ unspecified}$$

(1.2)

against the ordered alternatives

$$H_1 : \beta_1 \leqq \cdots \leqq \beta_k \text{(with at least one strict inequality)} \ ; \ \alpha_i \text{ unspecified.} \tag{1.3}$$

Parametric or nonparametric tests for the parallelism of regression lines against the ordered alternatives (1.3) have been considered by Adichie(1976) and Rao and Gore(1984), among others. Adichie(1976) proposed tests based on likelihood ratio statistic and a linear combination of maximum likelihood estimators of slopes, and rank analogues of these tests. Rao and Gore (1984) proposed distribution-free tests for the problem of testing the equality of intercepts (slopes) against the alternative that the interecepts(slopes) are in increasing(decreasing) order of magnitude. But they assumed special setting that the independent variables $x_{i.}$ are equispaced.

Recently, Jee(1989) proposed a nonparametric test which is asymptotically distribution-free. The asymptotic null distribution of the test statistic is the same as that of the Jonckheere test statistic in location problem. According to the Monte Carlo study, her test is proved to be very powerful. But, her test can be applied under the assumption of equal intercepts.

In this paper we consider a nonparametric test which is based on the Hodges-Lehmann type estimators of the slope parameters. We expect that the test is robust and reasonably powerful.

In Section2, the parametric and nonparametric tests proposed by Adichie(1976) are introduced. The test statistics are based on scores, which enable us to consider many alternative approaches by choosing different scores and slope estimators. According to the Monte Carlo study performed by Jee(1989), the procedure proposed by Rao and Gore(1984) does not appear to be better than the Adichie's tests. We therefore included only the Adichie's tests in this paper to compare with our proposed test.

Section 3 deals with the proposed test. The test statistic is a modified version of the Adichie's procedure based on scores. That is, the test statistic is a linear combination of the Hodges -Lehmann type estimators. The assumption of equal intercepts is not required in the proposed test.

Section 4 contains the results of a small-sample Monte Carlo study. In the case of $k=4$, a comparative study of the proposed test with the parametric and nonparametric tests was performed for several distributions. The results show that the proposed test is compatible with the Adichie's tests in medium-tailed distributions and more powerful than the Adichie's test in heavy-tailed distributions.

2. The Adichie's Tests Based on Scores

In this section we review the Adichie's tests based on scores. One of the tests is a linear combination of maximum likelihood estimators of β_i's and the other one is a rank version of this parametric test. To introduce the Adichie's tests, the following notations will be used. Let for $i=1,\cdots,k$,

$$\bar{x}_i=\sum_j x_{ij}/n_i \; ; \; \bar{Y}_i=\sum_j Y_{ij}/n_i,$$

$$w_i^2=\sum_j (x_{ij}-\bar{x}_i)^2 \; ; \; W^2=\sum_i w_i^2 \; ; \; r_i=w_i^2/W^2,$$

$$\tilde{\beta}_i=\sum_j (x_{ij}-\bar{x}_i)\, Y_{ij}/w_i^2 \; ; \; \tilde{\beta}=\sum_i r_i\tilde{\beta}_i \; ; \; N=\sum_i n_i. \tag{2.1}$$

The parametric test statistic based on scores, suggested by Adichie(1976), is of the form

$$S=\sum_i C_i\tilde{\beta}_i, \quad (\sum_i C_i=0). \tag{2.2}$$

When the scores C_i are nondecreasing, the test rejects H_0 for large values of S. Most of the following results are from Adichie(1976).

Assuming that F is normal, S has a normal distribution with mean $\sum_i C_i\beta_i$ and variance $\sigma^2\sum_i (C_i^2/w_i^2)$. If σ is assumed known, the power of S against any given $\beta_1\leqq\cdots\leqq\beta_k$ is given by

$$\Phi(\sum_i C_i\beta_i/\sigma(\sum_i(C_i^2/w_i^2))^{1/2}-u_a)=\Phi(\gamma\Delta/\sigma-u_a) \tag{2.3}$$

where u_a is the upper $100a\%$ point of the standard normal distribution, Δ^2 is the noncentrality parameter given by

$$\Delta^2=\sum_i w_i^2(\beta_i-\sum_i r_i\beta_i)^2, \tag{2.4}$$

and γ is defined by

$$\gamma=\sum_i C_i\beta_i/\Delta(\sum_i (C_i^2/w_i^2))^{1/2}. \quad (2.5)$$

Thus, when the alternative β_i's are preassigned, an optimum choice of scores can be given by

$$C_i=w_i^2(\beta_i-\sum_i r_i\beta_i). \quad (2.6)$$

But, the β_i's are not usually specified under alternatives. Only the orders of β_i's are available under alternatives. Thus we can not use the optimum scores in (2.6). To construct an increasing sequence of constants, Adichie (1976) suggested to use S_i instead of β_i, where S_i is defined by

$$S_i=w_1^2+\cdots+w_{i-1}^2+(w_i^2/2). \quad (2.7)$$

Substituting S_i for β_i in (2.6), the scores become

$$C_i=w_i^2(S_i-\sum_i r_iS_i). \quad (2.8)$$

Thus when σ^2 is not known, S can be studentized, to yield S_t which has a Student t distribution with $N-2k$ degrees of freedom. The statistic is defined as

$$S_t=(\sum_i (C_i^2/w_i^2))^{-1/2}\, S/\hat{\sigma} \quad (2.9)$$

where

$$\hat{\sigma}^2=\sum_i\sum_j (Y_{ij}-\bar{Y}_i-\hat{\beta}_i(x_{ij}-\bar{x}_i))^2/(N-2k).$$

As a rank version of the test statistic S in (2.2) Adichie (1976) proposed a rank test based on scores. The proposed test statistic is of the form

$$S_R=\sum_i C_i\hat{\beta}_{Ri}, \quad (\sum_i C_i=0), \quad (2.10)$$

where

$$\hat{\beta}_{Ri}=\{\sum_j (x_{ij}-\bar{x}_i)R_{ij}^*/(n_i+1)\}/w_i^2, \quad i=1,\cdots,k, \quad (2.11)$$

R_{ij}^* is the rank of the jth residual $Y_{ij}-\hat{\beta}(x_{ij}-\bar{x}_i)$ among the ith group of residuals with some estimator $\hat{\beta}$ of the common slope β. We again reject H_0 for large values of S_R.

To estimate the common slope β, Adichie (1976) defined a statistic which is equivalent to

$$G(\beta)=\sum_i\sum_j(x_{ij}-\bar{x}_i)R_{ij}/W^2, \tag{2.12}$$

where R_{ij} is the rank of $(Y_{ij}-\beta x_{ij})$ based on n_i observation within the ith group of samples. The proposed estimator $\hat{\beta}$ of the common slope β is defined by

$$\hat{\beta}=(\hat{\beta}_1+\hat{\beta}_2)/2 \tag{2.13}$$

where

$$\beta_1=\sup\{\beta : G(\beta)>0\},$$

$$\beta_2=\inf\{\beta : G(\beta)<0\}.$$

Adichie(1976) showed that, under H_0, $W\cdot S_R$ is asymptotically $N(0, \tau^2)$, where

$$\tau^2=(1/12)(\sum_i(C_i^2/r_i).$$

3. The Proposed Test Statistic and its Properties

In this section we propose a nonparametric test for testing the parallelism of regression lines against ordered alternatives. The proposed test statistic is based on a linear combination of robust slope estimators. That is, we want to construct a test statistic of the form

$$T=\sum_i C_i\hat{\beta}_i, \quad (\sum_i C_i=0), \tag{3.1}$$

where $\hat{\beta}_i$'s are robust estimators of slope parameters.

To obtain robust estimators of β_i's, we may use the Hodges-Lehmann type estimators proposed by Sen(1968) or Sievers(1978). In this paper we use the rank procedure suggested by Sievers(1978).

For each line i, without loss of generality, we assume that

$$x_{i1}\leqq x_{i2}\leqq\cdots\leqq x_{ini}$$

with at least one strict inequality. We define weighted rank statistics $U_i(\beta)$ by

$$U_i(\beta)=\sum_{s<t}a_{st}\Psi(Y_{it}-Y_{is}-\beta(x_{it}-x_{is})), \quad i=1,\cdots,k,$$

where Ψ is the indicator function defined by

$$\Psi(t)=\begin{cases}1, & t\geqq 0\\ 0, & t<0.\end{cases}$$

The weights a_{ist} are nonnegative and $a_{ist}=0$ whenever $x_{it}=x_{is}$. Let

$$a_{i..}=\sum_{s<t} a_{ist}, \quad i=1,\cdots,k.$$

Then, the Hodges-Lehmann type estimator of β_i based on $U_i(\beta)$ is defined by

$$\hat{\beta}_i=\frac{1}{2}(\beta_i^{*}+\beta_i^{**}), \tag{3.2}$$

where

$$\beta_i^{*}=\sup\{\beta : U_i(\beta)>a_{i..}/2\},$$

$$\beta_i^{**}=\inf\{\beta : U_i(\beta)<a_{i..}/2\}.$$

To obtain an explicit form of the estimator $\hat{\beta}_i$ in (3.2), we let

$$S_{ist}=\frac{Y_{it}-Y_{is}}{x_{it}-x_{is}}, \quad \text{for } x_{is}<x_{it}.$$

Then $U_i(\beta)$ is a function of S_{ist} since $\Psi(Y_{it}-Y_{is}-\beta(x_{it}-x_{is}))=1$ whenever $S_{ist}\geqq\beta$. We now consider the probability distribution defined on S_{ist} by assigning probability $a_{ist}/a_{i..}$ to S_{ist}. Then $\hat{\beta}_i$ in (3.2) is the median of this probability distribution, and it is called the weighted median estimator of β_i with weights a_{ist}.

For the weights $a_{ist}\equiv 1$, $\hat{\beta}_i$ is the median of S_{ist}, which is equivalent to the Sen's estimator. For the weights $a_{ist}=x_{it}-x_{is}$, $\hat{\beta}_i$ is the weighted median of S_{ist} with weights $x_{it}-x_{is}$, which was originally considered by Jaeckel (1972).

We now consider the asymptotic distribution of the proposed statistic T in (3.1) with the weighted median estimators $\hat{\beta}_i$'s. To apply the results in Sievers (1978), we let

$$A_{ij}=\sum_{s=1}^{j-1} a_{isj}-\sum_{t=j+1}^{n_i} a_{ijt},$$

and assume the following conditions :

As $\min_{1\leqq i\leqq k} n_i\rightarrow\infty$,

$C1$: $\sum_j A^2_{ij} / \max_{1 \le j \le n_i} A^2_{ij} \to \infty$,

$C2$: $\sum_{s<t} a^2_{ist} / \sum_j A^2_{ij} \to 0$,

$C3$: $\max_{1 \le j \le n_i} (x_{ij} - \bar{x}_i)^2 / n_i \to 0$,

$C4$: $\sum_j (x_{ij} - \bar{x}_i)^2 / n_i \to \sigma^2_{ix}, 0 < \sigma^2_{ix} < \infty$,

$C5$: $\sum_j A_{ij} (x_{ij} - \bar{x}_i) / \{\sum_j A^2_{ij} \cdot \sum_j (x_{ij} - \bar{x}_i)^2\}^{1/2} \to \rho_i \neq 0$.

From Theorem 5 of Sievers(1978) we have the following theorem.

Theorem 3.1. Let $T = \sum_i C_i \hat{\beta}_i$ (with $\sum C_i = 0$), where $\hat{\beta}_i$ is defined by(3.2). Assume that the observations $Y_{i1}, \cdots, Y_{in_i}$ have a joint density

$$\pi_j f(y_{ij} + n_i^{-1/2} \theta_i x_{ij}), \quad i = 1, \cdots, k,$$

where f is absolutely continuous with finite Fisher information. Then, under the conditions $C1 \sim C5$, T has asymptotically a normal distribution with mean $\sum_i C_i \beta_i$ and variance

$$\sum_i C_i^2 / \{12 n_i \rho_i^2 \sigma^2_{ix} (\int f^2 dx)^2\} \tag{3.3}$$

Remark. Note that ρ_i is the asymptotic correlation of the pairs (x_{ij}, A_{ij}), $j = 1, \cdots, n_i$. Thus, the asymptotic variance in(3.3) is minimized when $\rho_i = \pm 1$, which can be achieved with $a_{ist} = x_{it} - x_{is}$. We thus suggest to use the weights $a_{ist} = x_{it} - x_{is}$ in this paper.

To obtain the critical values of T by using normal approximation, we have to evaluate the values of asymptotic variance in(3.3) or equivalently $\int f^2 dx$, which depends on the underlying distribution. Assuming that F is normal, we have

$$\int f^2(x) dx = 1/(2\sqrt{\pi}\sigma) \tag{3.4}$$

We also need to estimate σ in(3.4). For example Song and Oh(1981) suggested to use

$$\hat{\sigma} = 1.48 \text{ med}\{|r_{ij}^*|\}, \tag{3.5}$$

where

$$r_{ij}^* = (r_{ij} - \hat{a}_i - \hat{\beta}_i x_{ij}) / \delta_{ij}$$

with

$$\hat{\delta}_{ij}^2=\frac{n_i-1}{n_i}\frac{(x_{ij}-\bar{x}_i)^2}{w_i^2}.$$

Thus, the asymptotic variance of T, with the optimum weights $a_{ist}=x_{it}-x_{is}$, can be approximated by

$$(\pi\hat{\sigma}^2/3)\sum_i C_i^2/w_i^2, \tag{3.6}$$

where $\hat{\sigma}$ is defined by (3.5). In the next section, the test statistic T is normalized using the approximate variance in (3.6).

We now consider the asymptotic relative efficiency (ARE) of the proposed test relative to the Adichie's test. According to Theorem 3.1, when we use the optimum weights $a_{ist}=x_{it}-x_{is}$, we have the following result :

$$\mathrm{ARE}(T, S_t)=12\sigma^2(\int f^2dx)^2,$$

which is the same as that of the Wilcoxon test relative to the t-test in location problem. From Theorem 4 of Adichie (1976) we also have the following result :

$$\mathrm{ARE}(T,\ S_R)=1.$$

4. Small Sample Monte Carlo Study

In this section we compare the small-sample behavior of the proposed test and Adichie's tests discussed in Section 2. Through a small-sample Monte Carlo study the empirical powers and significance levels were compared. The proposed test T in (3.1) was compared with the parametric test S_t in (2.9) and its rank version S_R in (2.10).

Each test used 4 regression lines ($k=4$) and an equal-sized sample ($n_i=10$). In all tests, the x's were fixed $(1, 2, \cdots, 10)$ and the Y's were obtained from

$$Y_{ij}=\alpha_i+\beta_i x_{ij}+\varepsilon_{ij},\ i=1, 2, 3, 4,\ \ j=1, \cdots, 10.$$

The ε_{ij}'s were randomly generated from various underlying distributions such as the uniform, normal, double exponential, Cauchy and contaminated normal distributions. The cdf of an ε-contaminated normal distribution is given by

$$F(x)=(1-\varepsilon)\Phi(x)-\varepsilon\Phi(x/\sigma), \tag{4.1}$$

where $\Phi(x)$ is the cdf of the standard normal. For the contaminated normal distributions, two cases were taken : ($\varepsilon=0.05$, $\sigma=3.0$) and ($\varepsilon=0.10$, $\sigma=5.0$) in(4.1). The random variates were generated from the IMSL subroutines on VAX 780 at Seoul National University. The uniform random variates were obtained by the subroutine GGUBT and the normal ones by the subroutine GGNPM. The inverse integral transformation was applied to generate double exponential and Cauchy random variates. For each distribution, the random variates were scaled to have a unit variance. Since the Cauchy distribution doesn't have the second moment, the random variates were divided by the value of $F^{-1}(0.84)-F^{-1}(0.5)=1.8326$. For the standard normal distribution the value of $\Phi^{-1}(0.84)-\Phi^{-1}(0.5)$ corresponds to the standard deviation.

We set $\alpha_i=1$ for convenience. For the choice of β_i under the ordered alternatives, we set the equally-spaced slopes given by

$$\beta_i=\beta_0+(i-1)m\Delta,$$

where m were chosen as 0, 1, 2, and β_0 was set 1 and Δ is the standard deviation of the least squares estimate of β for the combined sample.

1000 trials for each experimental situation were simulated and the number of rejections of H_0 at the nominal five percent level of significance was counted for each set. This procedure is an application of the methodology in Lo, Simkin and Worthley(1978). The results of the simulation study are presented in Table 1.

The results in Table 1 show that in extremely heavy-tailed distribution such as Cauchy the proposed test has higher empirical powers than the Adichie's parametric and nonparametric tests. In moderately heavy-tailed distributions such as contaminated normal and double exponential, the proposed test has slightly better powers than the Adichie's tests. In light-tailed distribution such as uniform, the proposed test has lower powers than the Adichie's test. In normal case the Adichie's parametric test is better than the others, and the proposed test is compatible with the Adichie's nonparametric test.

When $m=0$, the empirical powers indicate the empirical significance levels. The empirical significance levels of the proposed test in heavy-tailed distributions appear to be higher than the nominal level. We may thus investigate further to improve the asymptotic variance of T in the normalization procedure. We may also study further to find optimum scores to improve the power.

Table 1. Empirical Powers

($k=4$, $n_i=10$, $\alpha=0.05$, 1,000 replications)

distribution	*test*	$m=0$	$m=1$	$m=2$
Uniform	S_t	0.050	0.284	0.665
	S_R	0.037	0.253	0.575
	T	0.045	0.195	0.569
Normal	S_t	0.048	0.288	0.699
	S_R	0.051	0.239	0.638
	T	0.055	0.290	0.644
Double Exponential	S_t	0.063	0.304	0.761
	S_R	0.059	0.322	0.762
	T	0.077	0.416	0.783
Cauchy	S_t	0.047	0.101	0.194
	S_R	0.044	0.216	0.525
	T	0.083	0.340	0.653
Contaminated Normal ($\varepsilon=0.05$, $\sigma=3.0$)	S_t	0.047	0.293	0.726
	S_R	0.050	0.291	0.724
	T	0.055	0.347	0.736
Contaminated Normal ($\varepsilon=0.10$, $\sigma=5.0$)	S_t	0.046	0.330	0.738
	S_R	0.040	0.436	0.894
	T	0.063	0.535	0.907

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