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Decision Makes for the Reliability Apportionment in Subsystems with Different Effort Functions

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Summary and Conclusion

The optimal reliability apportionment of development effort to raise the reliability for systems of fixed structure is discussed from a deterministic standpoint and also when there is uncertainty in component reliabilities at various stages of development. The effort minimization algorithm requires that all subsystems be subject to the same effort function. A. Albert[1, 5] solved the problem for effort functions(series case) which were the same for all subsystems.

In this paper, the solution allows different effort functions to be associated with different subsystems and it accomodates the general structure of systems. In section 3-5, the effort functions under constraints subject to specified level of reliability are minimal and allow great flexibility in modeling the relationship between effort and reliability increasing. In view of the subjectivity involved in modeling the effort functions, and of the uncertainty in subsystem reliabilities, it is advisable to carry out the analysis with a variety of competing effort functions so as to assess the sensitivity of the results to the assumptions.

Particularly, the new factors, namely reliability importance of subsystem $i[T_i]$ and ratio of rate of increase of effort to reliability importance for subsystem $i[Q_i]$ were involved for analyzing of these problems in the study.

Two optimization programmings are suggested to determine the optimal reliability apportionment. Dynamic programming and Lagrange multiplier method may prove efficient in determining the reliability apportionment with minimum effort when the subsystems are subject to

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different but measurable effort functions. Sometimes we are forced to maximize the system reliability for a given total effort. This problem may also be formulated as an optimization problem under constraints and, when these constraints are appropriately defined. The Lagrangian approach can also be used to apportion the total effort available to subsystems so that system reliability is maximized. This problem is the dual of the minimization.

We believe that these methodologies for determination are useful aid to decision making concerning reliability improvement.

1. Introduction

The objective of reliability allocation is to use the reliability model to assign reliability to the subsystems so as to achieve a specified level of reliability goal for the system. This allocation problem is complex for several reasons, among which are : the role a subsystem plays for the functioning of the system, the method of accomplishing this function, the complexity of the subsystem, and the reliability of the subsystem chaning with the type of function to be performed. The problem is further complicated by the lack of detailed information on many of these factors early in the system design phase.

During the development course suppose that a system has not achieved a specified reliability goal. To meet this goal, we should determine the optimal apportionment to improve subsystem reliabilities so that the total resources allocation is a minimum. The quantification of resources is given in terms of effort functions which are as realistic and unrestrictive as possible.

The general effort minimization algorithm requires that all subsystems be subject to the same effort function. However, if this requirement cannot be met, we have to find a certain special methodology which may prove efficient in determining the reliability allocation with minimum expenditure of effort when the subsystems are subject to different but measurable effort functions.

This paper is to offer the optimal policy to determine the reliability allocation with minimum effort to increase the system reliability when the subsystems are subject to the different effort functions. Nonlinear programming using Lagrange Multipliers and Dynamic programming are employed to determine the optimal policy what we want to get.

2. Notation

m	: number of subsystem
P_K	: the state at stage K
R	: system reliability
R^*	: system reliability goal
x_i	: reliability level of subsystem i at the present state of development $0 \leq x_i \leq 1$

y_i	: reliability goal set for subsystem i, $x_i \le y_i \le 1$
y_i*	: optimal reliability goal apportioned to subsystem i that minimizes the total development effort.
Q_i	: ratio of rate of increase of effort to reliability importance for subsystem i
$f(x)$	: system reliability function for x
$f(y)$	: system reliability function for y
$G(x_i,\ y_i)$	: effort function associated with subsystem x_i, y_i
$G(R,\ R^*)$	: effort function for the system R, R^*
$T_i(\cdot)$	: reliability importance of subsystem i
$f_K(P_K)$	: optimal return function for stage K.

3. Effort Functions and Conditions

Let's suppose the required reliability R^* for the system is greater than the present reliability R. It is clear that to achieve $R^*>R$, the reliability of at least one subsystem must be increased. To do this, a certain amount of expenditure of "effort" is needed. Some typical examples of this effort are : further engineering development, extra manpower, extensive testing, new technology.

Let the effort function, denoted by $G(R_i,\ R_i^*)$, $i=1,2,\cdots,n$, be defined to be the amount of effort needed to increase the reliability of the ith subsystem from a level R_i to a new level R_i^*. The effort function $G(x,y)$, $y>x\ge 0$, is assumed to satisfy the following conditions :

1. $G(x_i,\ y_i)\begin{cases} >0, & y_i>x_i \\ =0, & \text{otherwise}\end{cases}$
2. $G(x_i,\ y_i)$ is nondecreasing in y_i for a fixed value of x_i and nonincreasing in y_i for a fixed value of y_i ; that is
   $$G(x_i,\ y_i) \le G(x_i,\ y_i+\Delta y_i),\quad \Delta y_i>0$$
   $$G(x_i,\ y_i) \ge G(x_i+\Delta x_i,\ y_i),\quad \Delta x_i>0$$
3. $G(x_i,\ Y_i)$ is additive, that is
   $$G(x_i,\ y_i)+G(y_i,\ z_i)=G(x_i,\ z_i),\quad x_i<y_i<z_i$$
4. $G(0,\ x_i)$ is differentiable in $(0,\ 1)$, $0<x_i<1$.
5. $d^2G(0,\ x_i)/d^2x_i>0$, $0\le x_i<1$
6. $G(x_i,\ y_i)\to\infty$ as $y_i\to 1$ for fixed x_i, $0\le x_i<1$
7. $G(R,R^*)=\sum_{i=1}^{m} G(x_i,\ y_i)$, $y_i\ge x_i$, with at least one inequality being strict.

A consequence of the conditions from 1 to 7 is that the problem has a solution within the m-dimensional unit hypercube. At the stage of development characterized by x the rate of increase of total effort with respect to system reliability is :

$$dG(0,R)/dR=\sum_{i=1}^{m}\{dG(0,\ x_i)/dx_i\}/\{\partial f(x)/\partial x_i\}$$

$$= \sum_{i=1}^{m} \{dG(0,\ x_i)/dx_i\}/T_i(x_i)$$

$$= \sum_{i=1}^{m} Q_i(x_i) \tag{1}$$

where $T_i = \partial f(x)/\partial x_i > 0$, $0 < x_i < 1$; $i = 1, 2, \cdots, m$. and $Q_i(x_i) = \{dG(0,\ x_i)/dx_i\}/T_i(x_i)$. T_i are called reliability importances and they feature prominently in the determination of the optimal reliability apportionment with minimal effort. Considering a series system for which $T_i(x_i) = R/x_i$.

If $x_1 < x_2 < \cdots < x_m$ then $T_1(x_1) > T_2(x_2) \cdots > T_m(x_m)$ and the component with highest failure probability is the most reliability important for the functioning of the system.

4. Optimal Reliability Apportionment

4.1 Lagrange Multiplier Approach

The problem may be formulated as nonlinear optimization under constraints.

$$\min \sum_{i=1}^{m} G(x_i,\ y_i) = G(R, R^*) \tag{2}$$

subject to

$$f(y_i) = R^*$$

and

$$0 \le x_i \le y_i \le 1, \quad i = 1, 2, \cdots, m$$

considering the Lagrange function(Assuming convex function ; Appendix)as :

$$L(y, \lambda) = \sum_{i=1}^{m} G(x_i,\ y_i) - \lambda[f(y_i) - R^*] \tag{3}$$

where R^* is the system reliability goal(specified reliability). In the above identity the initial reliabilities x_1, $x_2, \cdots, x_m$ are considered foxed.

$$\frac{\partial L(y, \lambda)}{\partial y_i} = \frac{\partial G(x_i,\ y_i)}{\partial y_i} + \lambda\left[\frac{\partial f(y_i)}{\partial y_i}\right]$$

$$=\frac{\partial G(x_i,\ y_i)}{\partial y_i}+\lambda T_i(y_i)=0 \tag{4}$$

$$\frac{\partial/(y,\ \lambda)}{\partial\lambda}=-(R^*-f(y_i))=0 \tag{5}$$

By additivity(condition 3) we may write $\partial G(x_i,\ y_i)/\partial y_i=dG(0,\ y_i)/dy_i$.

From equation(1), Let system reliability be R, and label the subsystems so that $Q_1(x_1)\leq Q_2(x_2)\leq\cdots\cdots\leq Q_m(x_m)$ be the corresponding values when system reliability is R^*, $R^*>R$. Then the optimality is :

$$Q_1(y_1)=Q_2(y_2)=\cdots\cdots=Q_K(y_K)\leq Q_{K+1}(y_{K+1})\leq\cdots\cdots\leq Q_m(y_m)$$

$$f(y)=R^*$$

K is determined by R^*-R, and $1\leq K\leq m$.

For a series system with all subsystem effort functions the same, the ordering $Q_1(x_1)\leq Q_2(x_2)\leq\cdots\cdots\leq Q_m(x_m)$ becomes $x_1\leq x_2\leq\cdots\cdots\leq x_m$ and the optimality is [1,2]

$$y_1=y_2=\cdots\cdots=y_K\leq y_{K+1}\leq\cdots\cdots\leq y_m$$

$$\prod_{i=K+1}^{m} y_i=y,\ \text{where}\ y_i=x_i,\ \text{for}\ i=K+1,\cdots,m.$$

The optimality is to increase the reliabilities of the K weakest subsystems to a common value and leave the remaining subsystems alone.

If we are required to solve the problem to maximize the system reliability for a given total effort, This problem may also be formulated as an optimization problem under constraints and, when these constraints are approprlately defined, Lagrange multidier method can be used to apportion the given total effort to subsystems so that system reliability is maximized.

$$\max\ R^*=f(y_i) \tag{6}$$

subject to

$$G(R,\ R^*)=\sum_{i=1}^{m}G(x_i,\ y_i)=E_c$$

and

$$C\leq x_i\leq y_i\leq 1,\ R^*>R$$

where Ec denote the amount of given total effort.

This optimization problem can be selved by a method similar to the one used in equation (3). We write the Lagrangian function (Assuming concave function, Appendix) as :

$$L(y, \lambda) = f(y_i) - \lambda\left(\sum_{i=1}^{m} G(x_i, y_i) - E_c\right) \tag{7}$$

The maximum is

$$\frac{\partial L(y, \lambda)}{\partial y_i} = \frac{\partial f(y_i)}{\partial y_i} + \lambda\left(\frac{\partial G(x_i, y_i)}{\partial y_i}\right) \tag{8}$$

$$= T_i(y_i) + \lambda\left(\frac{dG(0, y_i)}{dy_i}\right) = 0$$

$$\frac{\partial L(y, \lambda)}{\partial \lambda} = \left(\sum_{i=1}^{m} G(x_i, y_i)\right) - E_c = 0 \tag{9}$$

4.2 Dynamic Programming Approach

Let us consider a proposed system consisting of n subsystem, each of which is to be developed iudependently and are assumed to be in series. We wish to quantify the reliability goal to be set for each subsystem so that the system goal will be attained with a minimum expenditure of the development effort.

Defining the following optimization problem :

$$\min \sum_{i=1}^{n} G_i(x_i, y_i)$$

subject to

$$\prod_{i=1}^{m} y_i \geq R^* \tag{10}$$

and

$$0 \leq x_i \leq y_i \leq 1, \quad i = 1, 2, \cdots, n,$$

This problem can be converted to a dynamic prgramming problem in the fellowing fashion.

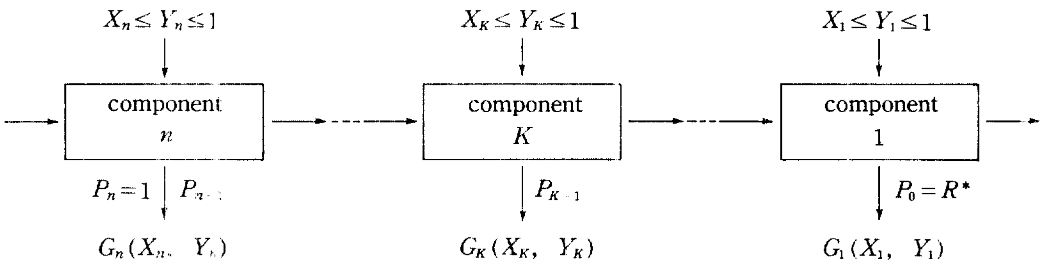
*** Stages**

Identify each of the in subsystems as a stage. Thus a reliability apportionment decision can be made sequentially at each stage.

*** State Variable**

Define the set P_K of all possible states P_K at stage K. Then we have the follawing relationship among the state variables :

$$1 = P_n \geq P_{n-1} \geq P_{n-2} \geq \cdots\cdots \geq P_1 \geq P_0 = \bar{y}$$



*** Transfer Relationships**

$$P_n Y_n = P_{n-1} \quad P_K Y_K = P_{K-1} \quad P_1 Y_1 = P_0$$

Let $\quad T_K(P_K, d_K) = P_K Y_K = P_{K-1} \quad K = 1, 2, \cdots, n$

The state variables indicate how much reliability may be allocated for the stage in order to meet the system reliability goal.

*** Decision Variable**

Define the set D_K of all possible decision alternatives $d_K = y_K$ at stage K such that $X_K \leq Y_K \leq 1$, $K = 1, 2, \cdots, m$.

*** Return Function**

$$R_K(P_K, d_K) = G_K(P_K, Y_K)$$
$$K = 1, 2, \cdots, m.$$

Let $f_K(P_K)$ be the optimal return function for stage K. Then the general recursion equation for the dynamic programming problem is

$$f_K(P_K) = \min_{d_K} \{G_K(P_K, d_K) + f_{K-1}(P_{K-1})\} \tag{11}$$

and $f_0(P_0)=0$. The above equation is used recursively backward to solve the problem.

Considering the parallel structure, its structure similar to series model. This is done by considering the unreliability levels of the subsystems. Let y_i be the reliability level for the ith subsystem. Then we want

$$1-\prod_{i=1}^{m}(1-y_i)\geq R^*$$

Let

$$W_i=1-y_i$$

Hence, the optimization problem is

$$\min \sum_{i=1}^{m} G_i\Big(x_i,\ \ (1-W_i)\Big)$$

subject to

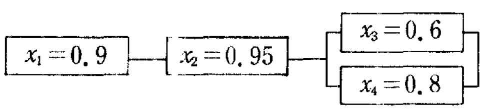
$$\prod_{i=1}^{m} W_i\leq 1-R^* \tag{12}$$

This problem presented by equation(12) is similar to the problem presented in equation(10) and hence can be solved by dynamic programming.

5. Numerical Example

Example 1 : Using Lagrangian Function

Considering a series-parallel system :



All subsystem states are mutually independent.
System reliability is :

$$\begin{aligned} R &= \Big(\prod_{i=1}^{2} x_i\Big)\Big(1-\prod_{i=3}^{4}(1-x_i)\Big) \\ &= x_1x_2(x_3+x_4-x_3x_4) \end{aligned} \tag{13}$$

and effort functions :

$$G(x_i, \quad y_i) = b_i(\ln(1-x_i)\ln(1-y_i))$$

$$i = 1, 2, \cdots, m \tag{14}$$

where b_i determine the relative effort between one subsystem and another. Let b_i denoted by $b_1 = 2$, $b_2 = 1$, $b_3 = 2$, $b_4 = 3$. The system reliability at the present is :

$$R = 0.7867$$

and $Q_i(x_i)$ are

$$Q_i(x_i) = \{dG(0, \quad x_i)/dx_i\}/T_i(x_i) \tag{15}$$

From condition 3 and 4 (additive and differentiable)

$$G(0, \quad x_i) = G(0, \quad y_i) - G(x_i, \quad y_i)$$

$$= b_i\left[\ln\frac{1}{1-x_i}\right]$$

hence,

$$\frac{dG(0, x_i)}{dx_i} = b_i\left[\frac{1}{1-x_i}\right], \text{ and } T_i(x_i) = \frac{\partial f(x_i)}{\partial x_i}$$

Then : $R = 0.7867$

$$Q_1(x_1) = 22.8, \quad Q_2(x_2) = 24.2, \quad Q_3(x_3) = 29.2, \quad Q_4(x_4) = 43.9$$

Since $Q_1(x_1) < Q_2(x_2) < Q_3(x_3) < Q_4(x_4)$, Relabelling is not necessary. Here, if we hope the system reliability would be increased to the value 0.95, we should determine the optimal reliability apportionment with minimum effort under specified system reliability 0.95.

In this example, the subsystem 4 has $b_4 = 3$, which is the coefficient of the highest relative effort of all the subsystems. Hence we let

$$Q_1(Y_1) = Q_2(y_2) = Q_3(y_3) < Q_4(y_4) \text{ and } y_4 = x_4$$

From equation (3), (4), (5) and (13), (14)

$$\frac{\partial L(y,\ \lambda)}{\partial y_1} = \left(\frac{b=2}{1-y_1}\right) + \lambda (y_2 y_3 + y_2 y_4 - y_2 y_3 y_4) = 0$$

$$\frac{\partial L(y,\ \lambda)}{\partial y_1} = \left(\frac{b_2=1}{1-y_2}\right) + \lambda (y_1 y_3 + y_1 y_4 - y_1 y_3 y_4) = 0$$

$$\frac{\partial L(y,\ \lambda)}{\partial y_3} = \left(\frac{b_3=2}{1-y_3}\right) + \lambda (y_1 y_2 - y_1 y_2 y_4) = 0$$

$$\frac{\partial L(y,\ \lambda)}{\partial y_4} = \left(\frac{b_4=3}{1-y_4}\right) + \lambda (y_1 y_2 - y_1 y_2 y_3) = 0$$

$$\frac{\partial L(y,\ \lambda)}{\partial \lambda} = y_1 y_2 y_3 + y_1 y_2 y_4 - y_1 y_2 y_3 y_4 - 0.95 = 0$$

The results are : $y_1^* = 0.98$, $y_2^* = 0.99$, $y_3^* = 0.90$, $y_4^* = 0.80$ and $Q_i^*(y_i) = \{dG(0,\ y_i)/dy_i\}/T_i(y_i)$, hence $Q_1^*(y_1) \fallingdotseq 103$, $Q_2^*(y_2) \fallingdotseq 103$, $Q_3^*(y_3) = 103$, $Q_4^*(y_4) \fallingdotseq 154$.

The minimum total effort for inereasing the system reliability up to 95% is :

$$G^*(R, R^*) = G^*(0.7866,\ 0.95) \equiv 2\ \ln 5 + \ln 5 + 2\ \ln 4 + 3\ \ln 1$$

$$\equiv 7.6$$

Example2 : A Dynamic Programming Approach

System has three independent subsystems. The system can function successfully if and only if each of the three subsystems functions properly. The system reliability requirement is 0.90. Based on engineering analysis and historical data for similar systems, the estimated reliability levels of the subsystems are 0.91, 0.92, 0.95. Now we should set the reliability goal for each subsystem in order to minimize the total expenditure of effort. The estimated effort functions are as shown in Table .

The recursion equations are :

$$f_1(P_1) = \min_{\{y_1 \geq 0.09/P_1\}} (G_1(0.91,\ y_1))$$

$$f_2(P_2) = \min_{\{y_2 \geq 0.90/P_2 y_1\}} (G_2(0.92,\ y_2) + f_1(P_1))$$

$$f_3(P_3) = \min_{\{y_3 \geq 0.90/P_3 y_2\}} (G_3(0.95,\ y_3) + f_2(P_2))$$

First we develop the state transformation tables that give us the set of all possible input states for various stages. Since the system reliability goal is 0.90

Table 1. Effort functions

y_1	$G_1(0.91,\ y_1)$	y_2	$G(0.92,\ y_2)$	y_3	$G(0.95,\ y_3)$
0.91	0				
0.92	1	0.92	0		
0.93	2	0.93	2		
0.94	3	0.94	4		
0.95	4.5	0.95	10	0.95	0
0.96	8	0.96	18	0.96	3
0.97	20	0.97	30	0.97	10
0.98	50	0.98	50	0.98	40
0.99	100	0.99	90	0.99	100

Table 2. Transformation at Stage 3

$T_3(P_3, y_3) = P_3 y_3 = 1 \cdot y_3 = 1 \cdot P_2$					
y_3	0.95	0.96	0.97	0.98	0.99
P_2	0.95	0.96	0.97	0.98	0.99

Table 3. Transformation at Stage 2

P_2 \ y_2	$P_1 = P_2 y_2$								
	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99
0.95	0.865	0.874	0.884	0.893	0.903	0.912	0.922	0.931	0.941
0.96	0.874	0.883	0.893	0.902	0.912	0.922	0.931	0.941	0.950
0.97	0.883	0.892	0.902	0.912	0.921	0.931	0.941	0.950	0.960
0.98	0.891	0.902	0.911	0.921	0.931	0.941	0.950	0.960	0.970
0.99	0.902	0.911	0.921	0.931	0.941	0.950	0.960	0.970	0.980

Values of P_1 less than 0.91 are not feasible and hence need not be considered. The infeasible values of the state variables are shown in Table 4 as blank entries. Only the values of P_0 greater than 0.90 are shown in Table 4.

Table 4. Transformation at Stage I.

P_1 \ y_1	$P_1 y_1 = P_0$							
	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99
0.911								0.902
0.912								0.903
0.921							0.902	0.912
0.922							0.904	0.913
0.931						0.903	0.912	0.922
0.941					0.903	0.912	0.921	0.932
0.950				0.903	0.912	0.921	0.931	0.941
0.960			0.902	0.912	0.921	0.931	0.941	0.951
0.970		0.902	0.912	0.921	0.931	0.941	0.951	0.960
0.980	0.902	0.911	0.921	0.931	0.941	0.951	0.960	0.970

Table 5.

P_1 \ y_1	$G_1(P_1,\ y_1)$								$f_1(P_1)$
	0.92	0.93	0.94	0.95	0.96*	0.97	0.98	0.99	
0.911								100	100
0.912								100	100
0.921							50	100	50
0.922							50	100	50
0.931						20	50	100	20
0.941					8*	20	50	100	8
0.950				4.5	8	20	50	100	4.5
0.960			3	4.5	8	20	50	100	3
0.970		2	3	4.5	8	20	50	100	2
0.980	1	2	3	4.5	8	20	50	100	1

We will now develop the optimal return functions. Using equation 11, we have

$$f_1(P_1) = \min_{y_1} \left[G_1(P_1,\ y_1) + f_0(P_0) \right]$$

$$= \min_{y_1} G_1(P_1,\ y_1)$$

since $f_0(P_0)$ is zero.

Proceeding to the next stage, we write

$$f_2(P_2) = \min_{y_2} \left[G_2(P_2,\ y_2) + f_1(P_1) \right]$$

Finally, we have

$$f_3(P_3) = \exp_{y_3} \left[G_3(P_3,\ y_3) + f_2(P_2) \right]$$

Table 7 shonws that the minimum value of the total effort is 48. Also, this is achieved when $y_3{}^* = 0.97$. Tracing backward, from Table 6, we find that $y_2{}^* = 0.97$ when $f_2(P_2) = 38$. This mean that $f_1(P_1) = 8$ and hence from Table 5 $y_1{}^* = 0.96$.

Table 6.

P_2 \ y_2	$Z_2(P_2,\ y_2) = G_2(P_2,\ y_2) + f(P_1)$							$f_2(P_2)$
	0.93	0.94	0.95	0.96	0.97*	0.98	0.99	
0.93						50+100	90+50	140
0.94					30+100	50+50	90+20	100
0.95				18+100	30+50	50+20	90+8	70
0.96			10+100	18+50	30+20	50+8	90+4.5	50
0.97		4+100	10+50	18+20	30+8*	50+4.5	90+3	38
0.98	2+100	4+50	10+20	18+8	30+4.5	50+3	90+2	26
0.99	2+50	4+20	10+8	18+4.5	30+3	50+2	90+1	18

Table 7.

P_3 \ y_3	$Z_3(P_3,\ y_3) = G_3(P_3,\ y_3) + f_2(P_2)$				$f_3(P_3)$
	0.96	0.97*	0.98	0.99	
1	3+50	10+38*	40+26	100+18	48

APPENDIX

1. Case of Minimization

$$G(R,\ R^*) = \sum G(x_i,\ y_i)$$

$$= b_1 \ln\left(\frac{1-x_1}{1-y_1}\right) + b_2 \ln\left(\frac{1-x_2}{1-y_2}\right)$$

$$+ b_3 \ln\left(\frac{1-x_3}{1-y_3}\right) + b_4 \ln\left(\frac{1-x_4}{1-y_4}\right)$$

$$f(y_i) = y_1y_2y_3 + y_1y_2y_4 - y_1y_2y_3y_4 - 0.95 = 0$$

$$L(y,\ \lambda) = \sum_{i=1}^{m} G(x_i,\ y_i) + \lambda(f(y_i) - R^*).$$

Determination of sufficient condition by using Hessian determinent as following :

let $\dfrac{\partial f(y_i)}{\partial y_i} = f_i$

and $L_{11} = 2/(1-y_1)^2,\ L_{22} = 1/(1-y_2)^2,\ L_{33} = 2/(1-y_3)^2,$

$L_{44} = 3/(1-y_4)^2,\ L_{12} = L_{21} = +\lambda(y_3 + y_4 - y_3y_4),$

$L_{13} = L_{31} = +\lambda(y_2 - y_2y_4),\ L_{14} = L_{41} = +\lambda(y_2 - y_2y_3),$

$L_{23} = L_{32} = +\lambda(y_1 - y_2y_4),\ L_{24} = L_{42} = +\lambda(y_1 - y_1y_3),$

$L_{34} = L_{43} = +\lambda(-y_1y_2)$

$$|\bar{H}| = \begin{vmatrix} 0 & f_1 & f_2 & f_3 & f_4 \\ f_1 & L_{11} & L_{12} & L_{13} & L_{14} \\ f_2 & L_{21} & L_{22} & L_{23} & L_{24} \\ f_3 & L_{31} & L_{32} & L_{33} & L_{34} \\ f_4 & L_{41} & L_{42} & L_{43} & L_{44} \end{vmatrix}$$

$$|\bar{H}_2| = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & L_{11} & L_{12} \\ f_2 & L_{21} & L_{22} \end{vmatrix} < 0, \quad |\bar{H}_3| = \begin{vmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & L_{11} & L_{12} & L_{13} \\ f_2 & L_{21} & L_{22} & L_{23} \\ f_3 & L_{31} & L_{32} & L_{33} \end{vmatrix} < 0$$

$$|\bar{H}_4| = |\bar{H}| < 0.$$

$G(R, R^*)$ is convex function with minimum value since $|\bar{H}_2| < 0$, $|\bar{H}_3| < 0$, $|\bar{H}_4| < 0$.

2. Case of Maximization

$$L(y, \lambda) = f(y_i) + \lambda\left(\sum_{i=1}^{m} G(x_i, y_i) - E_c\right)$$

Let $\dfrac{\partial G(x_i, y_i)}{\partial y_i} = g_i, \quad \dfrac{\partial^2 L(y, \lambda)}{\partial y_i \partial y_j} = L_{ij}$

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 & g_4 \\ g_1 & L_{11} & L_{12} & L_{13} & L_{14} \\ g_2 & L_{21} & L_{22} & L_{23} & L_{24} \\ g_3 & L_{31} & L_{32} & L_{33} & L_{34} \\ g_4 & L_{41} & L_{42} & L_{43} & L_{44} \end{vmatrix}$$

$$|\bar{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} > 0, \quad |\bar{H}_3| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & L_{11} & L_{12} & L_{13} \\ g_2 & L_{21} & L_{22} & L_{23} \\ g_3 & L_{31} & L_{32} & L_{33} \end{vmatrix} < 0$$

$$|\bar{H}_4| = |\bar{H}| > 0,$$

$f(y_i)$ is concave function with maximum value since $|\bar{H}_2| > 0$, $|\bar{H}_3| < 0$, $|\bar{H}_4| > 0$.

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