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CONDITIONS ON THE PROJECTIVE CURVATURE TENSOR OF CONFORMALLY FLAT RIEMANNIAN MANIFOLDS

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1. Introduction

In this paper we study conformally flat Riemannian manifolds satisfying one of the conditions $P \cdot Q = 0$, $Q \cdot P = 0$, $P \cdot R = 0$ $R \cdot P = 0$ and $P \cdot P = 0$, where R denotes the Riemann–Christoffel curvature tensor, P the Weyl projective curvature tensor and Q the Ricci endomorphism and where the first tensor acts on the second as a derivation. Riemannian manifolds and submanifolds satisfying similar conditions have been studied by various authors. For references, one can consult [3].

It was shown in [4] that each Riemannian manifold satisfying $R \cdot R = 0$ also satisfies $R \cdot P = 0$ and conversely. A classification of conformally flat spaces satisfying $R \cdot P = 0$ therefore reduces to a classification of conformally flat spaces satisfying $R \cdot R = 0$, which was done in [2] and [9].

Concering the conditions $P \cdot Q = 0$, $P \cdot P = 0$, $P \cdot R = 0$ and $Q \cdot P = 0$ we prove the following results.

Theorem 1. *Let (M^N, g) be a Riemann manifold for which $C = 0$, $(N \geq 3)$. Then the following assertions are equivalent:*
(i) (M^N, g) satisfies $P \cdot Q = 0$,
(ii) (M^N, g) satisfies $P \cdot P = 0$,
(iii) (M^N, g) satisfies $P \cdot R = 0$,
(iv) (M^N, g) is a space of constant curvature.

Theorem 2. *Let (M^N, g) be a Riemannian manifold for which $C = 0$, $(N \geq 3)$. Then, (M^N, g) satisfies $Q \cdot P = 0$ if and only if (M^N, g) has*

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constant curvature or is a simple conformally flat Riemannian manifold.

For simple (nearly) conformally flat Riemannian manifolds, see Sections 2 and 5.

2. Basic Formulas

Let (M^N, g) be a (connected) N–dimensional Riemannian manifold, $(N \geq 3)$. In the following, X, Y and Z denote vector fields on M^N, ∇ denotes the Levi Civita connection of (M^N, g), R the Riemann–Christoffel curvature tensor, Q the $(1,1)$–tensor related to the Ricci tensor S by $g(QX, Y) = S(X, Y)$ for all X and Y, $\tau = trQ$ the scalar curvature and, finally, $X \wedge Y$ denotes the $(1,1)$–tensor defined by $(X \wedge Y)Z := g(Z,Y)X - g(Z,X)Y$. Then, *Weyl's conformal curvature tensor* C and *Weyl's projective curvature tensor* P, are defined by

$$\begin{aligned} C(X,Y) \; := \; & R(X,Y) - \frac{1}{N-2}(QX \wedge Y + X \wedge QY) \\ & + \frac{\tau}{(N-1)(N-2)} X \wedge Y, \end{aligned} \tag{2.1}$$

and

$$P(X,Y) := R(X,Y) - \frac{1}{N-1}(X \wedge Y) \circ Q,$$

respectively.

(M^N, g) is called (locally) *conformally flat* if (M^N, g) is (locally) conformally equivalent to $\mathbf{E}^N$. For $N \geq 4$, (M^N, g) is conformally flat if and only if $C = 0$. We recall that every surface is conformally flat and that $C = 0$ for every 3–dimensional Riemannian manifold. Let D be the tensor defined by

$$\begin{aligned} D(X,Y,Z) \; := \; & (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\ & - \frac{1}{2(N-1)}((X \cdot \tau) g(Y,Z) - (Y \cdot \tau) g(X,Z)). \end{aligned} \tag{2.2}$$

If $N \geq 4$ and $C = 0$, then $D = 0$. For $N = 3$, (M^N, g) is conformally flat if and only if $D = 0$. In general, (M^N, g) will be called *nearly conformally flat* if $D = 0$. It is clear that conformally flat manifolds are nearly conformally flat. The divergence of a $(0,k)$–tensor T on (M, g) is the

$(0, k-1)$–tensor δT defined by $(\delta T)(X_2, \cdots, X_k) := -\sum_{i=1}^{N}(\nabla_{E_i}T)(E_i, X_2, \cdots, X_k)$, where $\{E_1, E_2, \cdots, E_N\}$ is a local orthonormal frame. A Riemannian manifold is said to have *harmonic curvature* (respectively *harmonic Weyl* (conformal) *curvature tensor*) if $\delta R = 0$ (respectively $\delta C = 0$), [1]. The second Bianchi identity implies that $(\delta R)(Z, X, Y) = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)$ and $(\delta C)(Z, X, Y) = \frac{n-3}{n-2}\, D(X, Y, Z)$. Now it is clear that manifolds with harmonic curvature are nearly conformally flat. Moreover, a Riemannian manifold is nearly conformally flat if and only if it has harmonic Weyl tensor in case $N \geq 4$ or if it is conformally flat in case $N = 3$. (M^N, g) is called (locally) *projectively flat* if around every point of M^N there exists a mapping to $\mathbf{E}^N$ preserving geodesics. For $N \geq 3$, (M^N, g) is projectively flat if and only if $P = 0$. Every surface satisfies $P = 0$. (M^N, g) will be called *simple* if rank $Q \leq 1$ everywhere on the manifold. (M^N, g) is *Einstein* if S is proportional to g. It is well known that every surface is Einstein and that an Einstein space satisfying $C = 0$ is a space of constant curvature.

Let $i : (M^N, g) \to (\tilde{M}^{N+1}, \tilde{g})$ be an isometric immersion. Let ξ be a local normal section on i. Then the *second fundamental tensor* A of i is defined by $\tilde{\nabla}_X \xi = -AX$, where $\tilde{\nabla}$ is the Levi Civita connection of $(\tilde{M}^{N+1}, \tilde{g})$. The curvature tensors R of (M^N, g) and $\tilde{R}$ of $(\tilde{M}^{N+1}, \tilde{g})$ are related by the equation of Gauss: $R(X, Y) = \tilde{R}(X, Y) + AX \wedge AY$. i is called *totally umbilical* if A is proportional to the identity map everywhere.

Let (B, g_B) and (F, g_F) be Riemannian manifolds and let $f : B \to \mathbf{R}_0^+$ be a ($C^\infty-$) function on B. Then, the *warped product* $(B, g_B) \times_f (F, g_F)$ is the Riemannian manifold $M := B \times F$ furnished with the metric tensor $g := \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F)$, where π is the projection $B \times F \to B$ onto the *base* and σ is the projection $B \times F \to F$ onto the *fiber*. Denote by X, Y and Z vector fields tangent to B and by U, V and W vector fields tangent to F. The *lift of a vector field* on B, say X, to M is the unique vector field of M that projects onto X and that is everywhere tangent to the *leaves*, i.e. the submanifolds of M of the form $B \times \{q\}$, where $q \in F$. We denote this new vector field also by X. The *lift of a vector field* on F is defined and denoted in the same way. The submanifolds of M of the form $\{p\} \times F$, where $p \in B$, are called *fibers*.

We will use the follwing formulas from [7], that express the Levi Civita connection ${}^M\nabla$, the Riemann–Christoffel curvature tensor ${}^M R$ and the

Ricci tensor M_S of (M,g) in terms of the Levi Civita connections B_∇ and F_∇, the Riemann–Christoffel curvature tensors B_R and F_R and the Ricci tensors B_S and F_S of (B,g_B) and (F,g_F) respectively:

$$
\begin{aligned}
&M_{\nabla_X Y} \quad \text{is the lift of} \quad B_{\nabla_X Y},\\
&M_{\nabla_X V} = M_{\nabla_V X} = \frac{X \cdot f}{f} V,\\
&\mathrm{nor} M_{\nabla_V W} = -\frac{g(V,W)}{f} grad\, f,\\
&\tan M_{\nabla_V W} \quad \text{is the lift of} \quad F_{\nabla_V W},\\
&M_{R(X,Y)Z} \quad \text{is the lift of} \quad B_{R(X,Y)Z},\\
&M_{R(V,X)Y} = -\frac{H^f(V,W)}{f} V,\\
&M_{R(X,Y)V} = M_{R(V,W)X} = 0, \qquad\qquad (2.3)\\
&M_{R(X,V)W} = -\frac{g(V,W)}{f} M_{\nabla_X (grad\, f)},\\
&M_{R(V,W)U} = F_{R(V,W)U} - \frac{g(grad\, f, grad\, f)}{f^2}(V \wedge W)U,\\
&M_{S(X,Y)} = B_{S(X,Y)} - \frac{d}{f} H^f(X,Y),\\
&M_{S(X,V)} = 0\\
&M_{S(V,W)} = F_{S(V,W) - g(V,W) f^\sharp},
\end{aligned}
$$

where tan is the projection of $T_{(p,q)}M$ onto $T_{(p,q)}(\{p\} \times F)$ and nor is the projection of $T_{(p,q)}M$ onto $T_{(p,q)}(B \times \{q\})$, and where H^f is the Hessian of f, $d = \dim F$ (assumed to be > 1) and where $f^\sharp = \frac{\Delta f}{f} + (d-1)\frac{g(grad\, f, grad\, f)}{f^2}$, Δf being the Laplacian of f on B.

Let (M^N, g) be a Riemannian manifold and let $p \in M^N$. In the following x, y and z denote vectors in T_pM. Let $x \wedge y$ denote the endomorphism $T_pM \to T_pM : z \mapsto g(z,y)x - g(z,x)y$. Since Q_p is symmetric, there exists an orthonormal basis $\{e_1, e_2, \cdots, e_N\}$ of (T_pM, g_p) consisting of eigenvectors of A_p, i.e. such that

$$Q_p e_i = \lambda_i e_i, \tag{2.4}$$

where $\lambda_i \in \mathbf{R}$ for each $i \in \{1, 2, \cdots, N\}$. If $N \geq 3$ and $C = 0$ on (M^N, g), then (2.1) and (2.4) imply that

$$R(e_i, e_j) = a_{ij} e_i \wedge e_j,$$

$$P(e_i, e_j)e_k = b_{ijk}(\delta_{jk}e_i - \delta_{ik}e_j),$$

where (2.5)

$$a_{ij} = \frac{(N-1)(\lambda_i + \lambda_j) - \tau}{(N-1)(N-2)},$$

$$b_{ijk} = \frac{(N-1)\lambda_i + (N-1)\lambda_j - (N-2)\lambda_k - \tau}{(N-1)(N-2)}$$

for all i, j and k in $\{1, 2, \cdots, N\}$.

According to [6] and [8] there exist N continuous functions $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ such that for each p in M the eigenvalues of Q_p are given by $\lambda_1(p), \lambda_2(p), \cdots, \lambda_N(p)$. Moreover, if for each p, Q_p has distinct eigenvalues $\bar{\lambda}_1(p) < \bar{\lambda}_2(p) < \cdots < \bar{\lambda}_r(p)$ with multiplicities $s_1, s_2, \cdots, s_r$ independent of p, then the functions $\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_r$ are differentiable. Then, for each point $p \in M$, there is an orthonormal frame $\{E_1, E_2, \cdots, E_N\}$ defined on a neighbourhood U of p such that all $E_i(q)$ are eigenvectors of $Q_q (q \in U)$.

Concerning the notations $P \cdot Q = 0, \cdots$ we say for example that (M^N, g) satisfies $P \cdot Q = 0$ if and only if $P(X, Y) \cdot Q = 0$ for all vector fields X and Y tangent to M^N, where $P(X, Y)$ acts as a derivation on the algebra of tensor fields on M^N, i.e.

$$(P(X, Y) \cdot Q)Z = P(X, Y)QZ - Q(P(X, Y)Z)$$

for X, Y, Z tangent to M. By e.g. $Q \cdot P = 0$ we express that

$$(Q \cdot P)(X, Y)Z := QP(X, Y)Z - P(QX, Y)Z - P(X, QY)Z - P(X, Y)QZ = 0$$

for all X, Y, Z tangent to M^N.

3. Proof of Theorem 1

The implications (iv)⇒(i), (iv)⇒(ii) and (iv)⇒ (iii) are trivial since $P = 0$ if (M^N, g) is a space of constant curvature. The implication (iii)⇒(i) follows from Lemma 2.1 (i) in [4]. Theorem 1 will thus be proven if the implications (i)⇒(iv) and (ii)⇒(iv) are shown, which we proceed to do next.

The implication (i)⇒(iv)

Suppose that (M^N, g) is a Riemannian manifold with $C = 0$ and satisfying $P \cdot Q = 0$. Take $p \in M$ and let $\{e_1, e_2, \cdots, e_N\}$ be a basis for T_pM

satisfying (2.4). Using the formulas (2.5), we find that $(P(e_i,e_j)\cdot Q)e_k = b_{ijk}\{\delta_{kj}(\lambda_k-\lambda_i)e_i - \delta_{ki}(\lambda_k-\lambda_j)e_j$ for all i,j and k in $\{1,2,\cdots,N\}$. From this it is easy to see that $P\cdot Q=0$ at p if and only if $(P(e_i,e_j)\cdot Q)e_i=0$ for all distinct i and j in $\{1,2,\cdots,N\}$, i.e. if and only if $b_{iji}(\lambda_j-\lambda_i)=0$ for all distinct i and j in $\{1,2,\cdots,N\}$. Interchanging i and j and subtracting yields that all λ_i are equal at p. It follows that (M^N,g) is Einstein. As (M^N,g) also has $C=0$, (M^N,g) is actually a space of constant curvature.

The implication (ii)⇒(iv)

Suppose that (M^N,g) is a Riemannian manifold with $C=0$ and satisfying $P\cdot P=0$. Take $p\in M^N$ and let $\{e_1,e_2,\cdots,e_N\}$ be a basis for T_pM^N satisfying (2.4). Using the formulas (2.5), we obtain from $(P(e_i,e_j)\cdot P)(e_i,e_j)e_k=0$ that $b_{iji}(b_{ikk}-b_{jkk})=0$ for all mutually distinct i,j and k in $\{1,2,\cdots,N\}$. Interchanging i and j and subtracting yields that all λ_i are equal at p. It follows that (M^N,g) is Einstein. Since also $C=0$, (M^N,g) again is a space of constant curvature.

4. The Condition $Q\cdot P=0$

Lemma 1. *Let (M^N,g) be a Riemannian manifold for which $C=0,(N\geq 3)$. Then the following assertions are equivalent:*

(i) *(M^N,g) satisfies $Q\cdot P=0$,*

(ii) *for each point p in M,Q_p has one of the following forms;*

(a) *λI_N with $\lambda\in\mathbf{R}_0$,*

(b) $\begin{pmatrix} \lambda & \vdots & \\ \cdots & \cdots & \cdots \\ & \vdots & 0_{N-1} \end{pmatrix}$ *with $\lambda\in\mathbf{R}$.*

Proof. Suppose that (M^N,g) is a Riemannian manifold with $C=0$. Take $p\in M$ and let $\{e_1,e_2,\cdots,e_N\}$ be a basis for T_pM satisfying (2.4). Using the formulas (2.5), we obtain that $(Q\cdot P)(e_i,e_j)e_k = b_{ijk}\{\delta_{ik}(\lambda_i+\lambda_k)e_j - \delta_{jk}(\lambda_j+\lambda_k)e_i\}$ for all i,j and k in $\{1,2,\cdots,N\}$. From this it is easy to see that $Q\cdot P=0$ at p if and only if $(Q\cdot P)(e_i,e_j)e_i=0$ for all mutually distinct i and j in $\{1,2,\cdots,N\}$, i.e. if and only if

$$\lambda_i b_{iji}=0 \tag{4.1}$$

for mutually distinct i and j in $\{1,2,\cdots,N\}$.

One of the implications in the lemma is now clear: if Q_p has one of the forms described in (a) and (b) in the lemma, then $Q \cdot P = 0$. Next, we assume that (M^N, g) satisfies $Q \cdot P = 0$. Interchanging i and j in (4.1), subtracting and making use of formula (2.5), then yields that

$$(\lambda_i - \lambda_j)(\lambda_i + \lambda_j - \tau) = 0. \tag{4.2}$$

Denote by $\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_r$ the mutually distinct eigenvalues of Q_p with multiplicities $s_1, s_2, \cdots, s_r$, respectively.
Suppose that $r \geq 3$. Choose mutually distinct indices α, β and γ in $\{1, 2, \cdots, r\}$. By (4.2), then $\bar{\lambda}_\alpha + \bar{\lambda}_\beta - \tau = 0$ and $\bar{\lambda}_\alpha + \bar{\lambda}_\gamma - \tau = 0$, which contradicts the fact that $\bar{\lambda}_\beta \neq \bar{\lambda}_\gamma$. Therefore, $r = 1$ or $r = 2$.
Suppose that $r = 2$. Taking $i = 1$ and $j = N$ in (4.1) and (4.2), one obtains that $(s_1 - 1)\bar{\lambda}_1 = 0$ and in the same way one finds that $(s_2 - 1)\bar{\lambda}_2 = 0$. Since $s_1 + s_2 \geq 3$, one of $\bar{\lambda}_1$ and $\bar{\lambda}_2$ is zero, say $\bar{\lambda}_2 = 0$. Then $s_1 = 1$ and hence Q_p has the form described in (b) in the lemma. Finally, if $r = 1$, then Q_p also has one of the desired forms.

Proof of Theorem 2
Suppose that (M^N, g) is a Riemannian manifold satisfying $C = 0$ and $Q \cdot P = 0$.

Assume that there is a point p in M such that Q_p takes the form described in (a) in the lemma. Call W the set of all such points and let W_0 be the connected component of W containing p. W_0 is Einstein and conformally flat and hence it has constant curvature. We will show that in this case $M = W_0$: since M is connected and W_0 is non–empty and open, it is sufficient to prove that W_0 is also closed. Suppose that $x \in \bar{W}_0$. Take a sequence $(x_n)_{n \in \mathrm{N}}$ in W_0 converging to x. λ is a constant on W_0. This gives, because of the continuity of the eigenvalue functions, that $\lambda_1(x) = \cdots = \lambda_N(x) = \lambda \neq 0$. Therefore $x \in W_0$.

If there is no point in M such that Q takes the form described in (a) in the lemma, then by definition and by the same lemma, M is a simple conformally flat Riemannian manifold. This proves one of the implications. The other one is trivial.

5. Simple Nearly Conformally Flat and Simple Conformally Flat Manifolds

In this section we determine the structure of simple nearly conformally flat manifolds and of simple conformally flat manifolds. First, we give examples and next we prove that these are essentially the only ones.

a. Examples

i. Suppose that g_F is an Einstein metric on an open part F of $\mathbf{R}^{N-1}$ in case $N > 3$, and in case $N = 3$ suppose that G_F is a metric of constant curvature on F. Denote the scalar curvature of (F, g_F) by r and let $f : B \to \mathbf{R}$ be a sloution of the differential equation

$$(N-2)(f')^2 + ff'' = \frac{r}{N-1} \tag{5.1}$$

which is nowhere zero, B being an open interval in $\mathbf{R}$. Call g_B the metric $(dx^1)^2$ on B. Then the warped product $(M, g) := (B, g_B) \times_f (F, g_F)$ is a simple nearly conformally flat manifold.

Indeed, easy computations using (2.3) and (5.1) show that

$$\begin{aligned} S(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}) &= -\frac{(n-1)f''}{f}, \\ S(\frac{\partial}{\partial x^1}, X) &= 0, \\ S(X, Y) &= 0, \quad \text{and} \\ D &= 0 \end{aligned} \tag{5.2}$$

(S in the Ricci tensor of (M, g), D is the tensor defined in (2.2) and X and Y are tangent to the fibers) and it is clear from (5.2) that rank $Q = 1$ everywhere if and only if f'' is nowhere zero.

ii. Keeping the same notations as before, if g_F is a metric of constant curvature and if $N \geq 4$, then (M, g) is a simple conformally flat manifold. The fact that (M, g) satisfies $C = 0$ follows easily from (2.1) and (2.3).

b. Theorem 3

We now prove the following result concerning the structure of simple nearly conformally flat manifolds.

Theorem 3. *Let (M^N, g) be a simple nearly conformally flat manifold and suppose that p is a point in (M^N, g) for which rank $Q_p = 1$. Then* *there exists a neighbourhood of p which is isometric to a manifold of the type described in 5.a.i.*

Proof. Suppose that (M^N, g) is a simple nearly conformally flat manifold. Let p be a point in M for which rank $Q_p = 1$. Since the eigenvalue functions are continuous and since by assumption rank $Q \leq 1$ everywhere on M, there exists a neighbourhood V_1 of p on which rank $Q = 1$ at every point. On V_1, Q has exactly two distinct eigenvalues and therefore there is a neighbourhood $V_2 \subset V_1$ of p on which there exists a differentiable function λ and a differentiable local orthonormal frame $\{E_1, E_2, \cdots, E_n\}$ such that $QE_1 = \lambda E_1$ and $QE_i = 0$ for all i in $\{2, 3, \cdots, N\}$. It is clear that on V_2 $\tau = \lambda$, where λ is the scalar curvature of (M, g). The eigenspaces of Q define (differentiable) distributions $T_1 = \{X \in TV_2 | QX = \lambda X\}$ and $T_2 = \{X \in TV_2 | QX = 0\}$. We use the fact that $D = 0$ under the form

$$(\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{2(N-1)}((X \cdot \tau)Y - (Y \cdot \tau)X) \tag{5.3}$$

to obtain information about these distributions. In the following, $X, X_1, X_2, \cdots$ denotes vector fields on V_2 with values in T_1 and $Y, Y_1, Y_2, \cdots$ denote vector fields on V_2 with values in T_2. Then (5.3) yields that $-Q[Y_1, Y_2] = \frac{1}{2(N-1)}((Y_1 \cdot \lambda)Y_2 - (Y_2 \cdot \lambda)Y_1)$. From this one easily concludes that T_2 is involutive and that λ is constant along the integral manifolds of T_2. Moreover, (5.3) implies that $\frac{1}{2(N-1)}(X \cdot \lambda)Y - (Q - \lambda)\nabla_Y X = -Q\nabla_X Y$. The left hand side is a vector field which has values in T_2 and the second one in T_1. Hence, both of them are zero. $Q\nabla_X Y = 0$ gives that

$$\nabla_X Y \quad \text{has values in} \quad T_2. \tag{5.4}$$

It is now easy to see that $\nabla_{X_1} X_2$ always has values in T_1. This implies that the integral curves of T_1 are geodesics. Denote by $(\nabla_Y X)_1$ and $(\nabla_Y X)_2$ the components of $\nabla_Y X$ in T_1, respectively in T_2. From $\frac{1}{2(N-1)}(X \cdot \lambda)Y - (Q - \lambda)\nabla_Y X = 0$, one obtains that

$$(\nabla_Y X)_2 = -\frac{1}{2(N-1)}(X \cdot \ln|\lambda|)Y. \tag{5.5}$$

Choose a system of coordinates $\varphi : U \to \mathbf{R}^N$ on a neighbourhood $U \subset V_2$ of p with coordinate functions $x_1, x_2, \cdots, x_n$ such that the integral manifolds of T_1 are given by the equation

$$\begin{cases} x_2 &= a_2 \\ x_3 &= a_3 \\ & \vdots \\ x_N &= a_N \end{cases}$$

and those of T_2 by

$$x_1 = a_1$$

$(a_1, a_2, \cdots, a_N \in \mathbf{R})$ and such that $\varphi(U) = W_1 \times W_2 \subset \mathbf{R} \times \mathbf{R}^{N-1}$, where W_1 and W_2 are open rectangles in $\mathbf{R}$, respectively in $\mathbf{R}^{N-1}$. Then $\frac{\partial}{\partial x_1}$ has values in T_1 and $\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \cdots, \frac{\partial}{\partial x_N}$ have values in T_2. λ (more precisely; its coordinate expression) is a function of x_1 only. We calculate $\frac{\partial}{\partial x_i} g(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1})$ for all $i \in \{2, 3, \cdots, N\}$

$$\begin{aligned} \frac{\partial}{\partial x_i} g(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}) &= 2g(\frac{\partial}{\partial x_1}, \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_1}) \\ &= 2g(\frac{\partial}{\partial x_1}, \nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_i}) \\ &= 0, \end{aligned}$$

because $\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_i}$ has values in T_2 (see (5.4)). We may therefore assume that $g(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}) = 1$, i.e. that $E_1 = \frac{\partial}{\partial x_1}$ and that

$$g = (dx_1)^2 + \sum_{i,j=2}^{N} g_{ij} dx_i dx_j.$$

Let q be a point in U. We already know that the integral manifold of T_1 through q is a geodesic. We study the integral manifold $M_2(q)$ of T_2 passing through q and the inclusion $i_2 : M_2(q) \hookrightarrow M$. Denote by $\nabla_2, R_2, S_2, \cdots$ the connection, the curvature tensor, $\cdots$ of $M_2(q)$. $E_2, E_3, \cdots, E_N$ is a local orthonormal frame for $TM_2(q)$ and E_1 is a normal vector field on i_2. (5.5) means that i_2 is totally umbilical :

$$(\nabla_{E_i} E_1)_2 = (\frac{\partial}{\partial x_1} \ln |\lambda|^{\frac{-1}{2(N-1)}}) E_i. \tag{5.6}$$

Denote by A the second fundamental tensor of i_2 with respect to the normal E_1, call $f = |\lambda|^{\frac{-1}{2(N-1)}}$ and $\alpha = -\frac{\partial}{\partial x_1} \ln f$. Then

$$A = \alpha I. \tag{5.7}$$

For every i in $\{2,3,\cdots,N\}$

$$
\begin{aligned}
Q_2E_i &= \sum_{j=2}^{N} R_2(E_i,E_j)E_j \\
&= \sum_{j=2}^{N}(R(E_i,E_j)E_j + (AE_i \wedge AE_j)E_j) \qquad (5.8)\\
&= \sum_{j=1}^{N} R(E_i,E_j)E_j - R(E_i,E_1)E_1 \\
&\quad + \sum_{j=2}^{N} g(E_j,AE_j)AE_i - \sum_{j=2}^{N} g(E_j,AE_i)AE_j \\
&= QE_i - R(E_i,E_1)E_1 + (N-1)\alpha^2E_i - \alpha^2E_i \\
&= ((N-1)\alpha^2 - \frac{\partial\alpha}{\partial x_1})E_i.
\end{aligned}
$$

In the last step we used the fact that

$$R(E_i,E_1)E_1 = \nabla_{E_i}\nabla_{E_1}E_1 - \nabla_{E_1}\nabla_{E_i}E_1 - \nabla_{[E_i,E_1]}E_1 = (\frac{\partial\alpha}{\partial x_1} - \alpha^2)E_i$$

(use (5.6)). (5.8) shows that $M_2(q)$ is an Einstein manifold :

$$Q_2 = \beta I, \qquad (5.9)$$

where

$$\beta = (N-1)\alpha^2 - \frac{\partial\alpha}{\partial x_1}.$$

If $N = 3$, it follows from (5.9) and the fact that λ is a funciton of x_1 only that $M_2(q)$ has constant curvature.

Now we show that all $M_2(q)$ are homothetic. First, using (5.6) one obtains that

$$\frac{\partial g_{ij}}{\partial x_1} = -2\alpha g_{ij}$$

for all $i,j \in \{2,3,\cdots,N\}$.

Let $g^* := f^{-2}g$. Then

$$\frac{\partial g^*_{ij}}{\partial x_1} = 0$$

for all $i,j \in \{2,3,\cdots,N\}$.

It is clear that g^* determines a metric on W_2, which we also denote by g^*. This metric is also an Einstein metric since it is homothetic to the restriction of g to $M_2(q)$. Denote the scalar curvature of g^* by r. Then

$$(N-1)\beta = \frac{r}{f^2} \tag{5.10}$$

It is clear that f determines a function on W_1, which we also denote by f. From (5.10), it follows that

$$(N-2)(f')^2 + ff'' = \frac{r}{N-1}.$$

Furthermore, f is nowhere zero. It is now proven that (U, g) is isometric to a manifold of the type described in 5.a.i.

c. Theorem 4

In this section we prove the following results concerning the structure of simple conformally flat manifolds.

Theorem 4. *Let (M^N, g) be a simple conformally flat manifold $(N \geq 4)$ and suppose that p is a point in (M^N, g) for which rank $Q_p = 1$. Then there exists a neighbourhood of p which is isometric to a manifold of the type described in* 5.a.ii.

Proof. Suppose that (M^N, g) is a simple conformally flat manifold. Let p be a point in M for which rank $Q_p = 1$. By Theorem 3 there exists a neighbourhood of p which is isometric to a manifold of the type described in 5.a.i. We show that the restriction of g to each $M_2(q)$ has constant curvature. In fact,

$$\begin{aligned} R_2(X,Y)Z &= R(X,Y)Z + (AX \wedge AY)Z \\ &= \left(\frac{QX \wedge Y + X \wedge QY}{N-2} - \frac{\tau X \wedge Y}{(N-1)(N-2)} + AX \wedge AY\right)Z \\ &= \left(-\frac{\tau}{(N-1)(N-2)} + \alpha^2\right)(X \wedge Y)Z \end{aligned}$$

for all X, Y and Z tangent to $M_2(q)$.
This terminates the proof.

d. Some Corollaries

Corollary 1. *Every simple nearly conformally flat manifold with constant scalar curvature is Ricci–flat.*

Proof. Suppose that (M^N, g) is a simple nearly conformally flat manifold with non–zero constant scalar curvature. We will deduce a contradiction. From Theorem 3 it follows that we can restrict ourselves to the manifolds described in 5.a.i. Since the scalar curvature is non–zero, f'' is nowhere zero (see (5.2)). Moreover, since the scalar curvature is constant, it follows from (5.2) that $f'f'' = \cdot f'''f$. Differentiating (5.1) gives that $f'f'' = 0$. Since f'' is nowhere zero, it follows that $f' = 0$. Hence $f'' = 0$, which gives the desired contradiction.

It is well known that there exist Einstein metrics which are not of cons–tant curvature. Hence, in view of Corollary 1, Theorem 3 and Theorem 4, we get the following result.

Corollary 2. *For each $N \geq 4$, there exist N-dimensional non-conformally flat simple nearly conformally flat manifolds which are not of harmonic curvature.*

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