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SEPARATION AXIOMS BETWEEN $T_{\frac{1}{2}}$ AND T_0

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1. Introduction

The well known axioms T_F, T_D, T_{UD} and T_O admit characterizations of the form "for arbitrary points x, y of the space, if y belongs to the derived set of $\{x\}$ then $\cdots$". We present here all the axioms obtained by postulating, in the consequent of the above conditional expression, the enunciative forms that may be constructed from conjunctions ($\wedge$) and disjuntions ($\vee$) of the following statements concerning the derived sets of $\{x\}$ and $\{y\}$ the derived set of $\{x\}$ is a closed set ("$d\{x\}cs$"), or it is a union of disjoint closed sets ("$d\{x\}udcs$") or it is a union of closed sets ("$d\{x\}ucs$") ; and the derived set of $\{y\}$ is empty ("$d\{y\} = \emptyset$"), or it is closed ("$d\{y\}cs$"), or it is a union of disjoint closed sets ("$d\{y\}udcs$"), or it is a union of closed sets ("$d\{y\}ucs$").

On the other hand, it is known that the shortest characterization, in terms of derived sets of points, of T_D-spaces (i.e. "for every point x of the space, $d\{x\}cs$") and that of T_O-spaces (i.e. "for every point x of the space, $d\{x\}ucs$") remains valid if we substitute the arbitrary point x by an arbitrary subset A of the space. This propriety is also true for all the statements of T_D and T_O presented here; we express this fact by $T_D^A = T_D$ and $T_O^A = T_O$. However, this is not valid for all the other axioms treated in this paper, that is, $T_\alpha^A \neq T_\alpha$ ($\alpha = FD, F, HG, HUD, H, G, FUD, IUH, I, UD, KUH, KJ, K, UH, J$ and UI). Nevertheless, we study some axioms of the form T_α^A and obtain new characterizations of $T_{\frac{1}{2}}$-spaces.

2. New separation axioms

Proposition 2.1. *A topological space (X, τ) is a T_D-space iff one of the following conditions holds:*

1) *for every* $x \in X$, $d\{x\}cs$. [1]
2) *for arbitrary* $x, y \in X$, $y \in d\{x\}$ *implies* $d\{x\}cs$.
3) *for arbitrary* $x, y \in X$, $y \in d\{x\}$ *implies* $(d\{x\}cs \wedge d\{y\}cs)$.
4) *for arbitrary* $x, y \in X$, $y \in d\{x\}$ *implies* $(d\{x\}cs \wedge d\{y\}udcs)$.

Proof. Implications ($1\Rightarrow 2$), ($3\Rightarrow 4$) and ($4\Rightarrow 1$) are immediate.
($2\Rightarrow 3$) Let x belong to X. If $d\{x\} = \emptyset$, then condition 3 holds. If $d\{x\} \neq \emptyset$, then $d\{x\}$ is a closed set and, for every point y in $d\{x\}$, if $d\{y\}$ is not empty it is a closed set from condition 2.

Proposition 2.2. *A topological space* (X, τ) *is a* T_{UD}*-space iff one the following conditions holds :*
1) *for every* $x \in X$, $d\{x\}udcs$. [1]
2) *for arbitrary* $x, y \in X$, $y \in d\{x\}$ *implies* $d\{x\}udcs$.
3) *for arbitrary* $x, y \in X$, $y \in d\{x\}$ *implies* $(d\{x\}udcs \vee d\{y\} = \emptyset)$.
4) *for arbitrary* $x, y \in X$, $y \in d\{x\}$ *implies* $(d\{x\}udcs \wedge d\{y\}udcs)$.
5) *for arbitrary* $x, y \in X$, $y \in d\{x\}$ *implies* $[d\{x\}cs \vee (d\{x\}udcs \wedge d\{y\}udcs)]$.
6) *for arbitrary* $x, y \in X$, $y \in d\{x\}$ *implies* $[(d\{x\}udcs \vee d\{y\} = \emptyset) \wedge d\{y\}udcs)]$.
7) *for arbitrary* $x, y \in X$, $y \in d\{x\}$ *implies* $[(d\{x\}cs \vee d\{y\}udcs) \wedge (d\{x\}udcs \vee d\{y\} = \emptyset)]$.

Proof. It is immediate.

Definition 2.3. A topological space (X, τ) will be called a T_H–space if for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $d\{y\}cs$.

Definition 2.4. A topological space (X, τ) will be called a T_{UH}–space if for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $d\{y\}udcs$.

Definition 2.5. A topological space (X, τ) will be called a T_I–space if for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $(d\{x\}cs \vee d\{y\}cs)$.

Definition 2.6. A topological space (X, τ) will be called a T_{UI}–space if for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $(d\{x\}udcs \vee d\{y\}udcs)$.

Proposition 2.7. *A topological space* (X, τ) *is a* T_F*–space iff one of the following condition holds:*
1) *for arbitrary* $x, y \in X$, $y \in d\{x\}$ *implies* $d\{y\} = \emptyset$. [1]
2) *for arbitrary* $x, y \in X$, $y \in d\{x\}$ *implies* $(d\{x\}udcs \wedge d\{y\} = \emptyset)$.

Proof. Implication ($2\Rightarrow 1$) is immediate.

(1⇒ 2) Let x belong to X. If $d\{x\} = \emptyset$, then condition 2 holds. If $d\{x\} \neq \emptyset$, then for every y in $d\{x\}, d\{y\} = \emptyset$ as a consequence of condition 1 and consequently $d\{x\}$ consists of closed points.

Proposition 2.8. *In a topological space (X, τ) the following conditions are equivalent and, if they hold, the space will be called a T_G–space:*

1) *for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $(d\{x\}cs \vee d\{y\} = \emptyset)$.*

2) *for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $[(d\{x\}cs \wedge d\{y\}udcs) \vee d\{y\} = \emptyset]$.*

3) *for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $[d\{x\}cs \vee (d\{x\}udcs \wedge d\{y\} = \emptyset)]$.*

4) *for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $[(d\{x\}cs \wedge d\{y\}udcs) \vee (d\{x\}udcs \wedge d\{y\} = \emptyset)]$.*

Proof. Implications (4⇒ 3) and (2⇒ 1) are immediate.
(3⇒ 2) Let x belong to X. If $d\{x\} = \emptyset$, then condition 2 holds. If $d\{x\} \neq \emptyset$ and, for every point y in $d\{x\}$, $d\{y\} = \emptyset$, then condition 2 is true. If there is a point y in $d\{x\}$ such that $d\{y\} \neq \emptyset$, then $d\{x\}$ is a closed set and $d\{y\}$ is a union of disjoint closed sets.
(1⇒4) Let x belong to X. If $d\{x\} = \emptyset$, then condition 4 holds. If $d\{x\} \neq \emptyset$ and for every point y in $d\{x\}$, $d\{y\} = \emptyset$, then $d\{x\}$ is a union of closed points and consequently condition 4 is true. If there exists a point y in $d\{x\}$ such that $d\{y\} \neq \emptyset$, then $d\{x\}$ is a closed set and either $d\{y\}$ is a closed set or it is a union of closed points.

Definition 2.9. A topological space (X, τ) will be called a T_K–space if for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $(d\{x\}udcs \vee d\{y\}cs)$.

Definition 2.10. A topological space (X, τ) will be called a T_J–space if for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $(d\{x\}cs \vee d\{y\}udcs)$.

Definition 2.11. A topological space (X, τ) will be called a T_{FD}–space if for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $(d\{x\}cs \wedge d\{y\} = \emptyset)$.

Proposition 2.12. *In a topological space (X, τ) the following conditions are equivalent and, if they hold, the space will be called a T_{HG}–space:*

1) *for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $[(d\{x\}cs \wedge d\{y\}cs) \vee d\{y\} = \emptyset]$.*

2) *for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $[(d\{x\}cs \wedge d\{y\}cs) \vee (d\{x\}udcs \wedge d\{y\} = \emptyset)]$.*

Proof. Implication (2⇒1) is immediate.

(1⇒2) Let x belong to X. If $d\{x\} = \emptyset$, then condition 2 holds. If $d\{x\} \neq \emptyset$ and for every point y in $d\{x\}, d\{y\} = \emptyset$, then $d\{x\}$ is a union of closed points and consequently condition 2 is true. If there exists a point y in $d\{x\}$ such that $d\{y\} \neq \emptyset$, then $d\{x\}$ is a closed set and $d\{y\}$ is a closed set as a consequence of condition 1.

Proposition 2.13. *In a topological space (X, τ) the following conditions are equivalent and, if they hold, the space will be called a T_{HUD}-space:*

1) *for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $(d\{x\}udcs \wedge d\{y\}cs)$.*

2) *for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $[(d\{x\}udcs \wedge d\{y\}cs) \vee d\{y\} = \emptyset]$.*

Proof. Implication (1⇒2) is immediate.

(2⇒1) Let x belong to X. If $d\{x\} = \emptyset$, then condition 1 holds. If $d\{x\} \neq \emptyset$ and for every point y in $d\{x\}, d\{y\} = \emptyset$, then $d\{x\}$ is a union of closed points. If there exists a point y in $d\{x\}$ such that $d\{y\} \neq \emptyset$, then $d\{x\}$ is a union of disjoint closed sets and $d\{y\}$ is a closed set as a consequence of condition 2.

Proposition 2.14. *In a topological space (X, τ) the following conditions are equivalent and, if they hold, the space will be called a T_{IUD}-space:*

1) *for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $[d\{x\}cs \vee (d\{x\}udcs \wedge d\{y\}cs)]$.*

2) *for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $[(d\{x\}cs \wedge d\{y\}udcs) \vee (d\{x\}udcs \wedge d\{y\}cs)]$.*

3) *for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $[(d\{x\}cs \vee d\{y\}cs) \wedge (d\{x\}udcs \vee d\{y\} = \emptyset)]$.*

4) *for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $[(d\{x\}cs \wedge d\{y\}udcs) \vee (d\{x\}udcs \wedge d\{y\}cs) \vee d\{y\} = \emptyset)]$.*

Proof. Implication (2⇒1) is immediate.

(4⇒3) Let x belong to X. If $d\{x\} = \emptyset$, then condition 3 holds. Let y be a point in $d\{x\}$, if $d\{y\} = \emptyset$, then condition 3 is true and if $d\{y\} \neq \emptyset$, then either $(d\{x\}cs \wedge d\{y\}udcs)$ or $(d\{x\}udcs\wedge d\{y\}cs)$; in any case condition 3 is verified.

(3⇒2) Condition 3 implies trivially that the derived set, $d\{x\}$, of every point x in X is a union of disjoint closed sets.

Let x belong to X. If $d\{x\} = \emptyset$, then condition 2 is true. If y is a point in $d\{x\}$, both $d\{x\}$ and $d\{y\}$ are unions of disjoint closed sets and, by condition 3, either $d\{x\}$ or $d\{y\}$ is a closed set, therefore condition 2 holds.

(1⇒4) Let x belong to X. If $d\{x\}$ is not a closed set then, it is a union of disjoint closed sets and, for every point y in $d\{x\}, d\{y\}$ is a closed set. If $d\{x\}$ is a non–empty closed set, for every point y in $d\{x\}$, $d\{y\}$ is a closed set or it is a union of disjoint closed sets by condition 1. Therefore in any case condition 4 is verified.

Definition 2.15. A topological space (X, τ) will be called a T_{IUH}–space if for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $[(d\{x\}cs \wedge d\{y\}udcs) \vee d\{y\}cs]$.

Definition 2.16. A topological space (X, τ) will be called a T_{KUH}–space if for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $[(d\{x\}udcs \wedge d\{y\}udcs) \vee d\{y\}cs]$.

Definition 2.17. A topological space (X, τ) will be called a T_{KJ}–space if for arbitrary $x, y \in X$, $y \in d\{x\}$ implies $[(d\{x\}cs \vee d\{y\}udcs) \wedge (d\{x\}udcs \vee d\{y\}cs)]$.

Proposition 2.18. *A topological space (X, τ) is a T_0-space iff one of the following conditions holds :*

1) *for every $x \in X$, $d\{x\}ucs$.* [1]
2) *for arbitrary $x, y \in X, y \in d\{x\}$ implies $d\{x\}ucs$.*
3) *for arbitrary $x, y \in X, y \in d\{x\}$ implies $d\{y\}ucs$.*

Proof. Implications (1⇒ 2) and (2⇒3) are immediate.

(3⇒1) Let x belong to X. If $d\{x\}$ is not a union of closed sets, there is a point y in $d\{x\}$ such that $\{\bar{y}\} = \{\bar{x}\}$. Therefore x belongs to $d\{y\}$ and, since $d\{y\}$ is a union of closed sets from the hypothesis, $d\{x\} \subseteq d\{y\}$ against the fact that $y \in d\{x\}$. .newpage

3. Relations among these axioms

Proposition 3.1.

$$
\begin{tikzcd}
T_{FD} \arrow[r] \arrow[d] & T_D \arrow[d] & & & & \\
T_F \arrow[r] & T_{HG} \arrow[r] \arrow[d] & T_G \arrow[d] & & & \\
& T_{HUD} \arrow[r] \arrow[d] & T_{IUD} \arrow[r] \arrow[d] & T_{UD} \arrow[d] & & \\
& T_H \arrow[r] & T_{IUH} \arrow[r] \arrow[d] & T_{KUH} \arrow[r] \arrow[d] & T_{UH} \arrow[d] & \\
& & T_I \arrow[r] & T_{KJ} \arrow[r] \arrow[d] & T_J \arrow[d] & \\
& & & T_K \arrow[r] & T_{UI} \arrow[r] & T_0
\end{tikzcd}
$$

Proof. It becomes immediate by comparing their respective chracterizations.

We give now a list of examples to prove that all these classes of topological spaces are different. In all the examples X is the set of real numbers and it is understood that the null set and the set X are closed.

Example 3.2. Let the sets $[x, +\infty[(x \in \mathbf{R})$ and $]x, +\infty[(x \in \mathbf{R})$ be closed. This space is T_D but not T_F.

Example 3.3. Let the sets $\mathbf{R}\backslash\{0\}, \{x\}$ $(x \neq 0; x \neq 1)$ and their finite unions, be closed. This space is T_G but not T_H.

Example 3.4. Let the sets $\{x\}$ $(x \in \mathbf{R}\backslash\ \mathbf{Z}),]x-1, x]$ $(x \in \mathbf{Z}; x > 0)$, $[x, x+1[(x \in \mathbf{Z}; x < 0)$ and their finite unions, be closed. This space is T_{UD} but not T_I.

Example 3.5. Let the sets $[-1,1]$, $\{x\}(-1 \leq x \leq 1; x \neq 0)$, $[-1,1] \cup \{x\}$ $(x \neq 2)$ and their finite unions, be closed. This space is T_{UH} but not T_K.

Example 3.6. Let the sets $[x, +\infty[(x \in \mathbf{R})$ be closed. This space is T_0 but not T_{UI}.

Example 3.7. Let the sets $\{x\}$ $(x \neq 0)$ and their finite unions be closed. This space is T_F but not T_D.

Example 3.8. Let the sets $\{x\}(x > 0)$, $\{-x, x\}(x > 0)$ and their finite unions, be closed. This space is T_{HUD} but not T_G.

Example 3.9. Let the sets $[x, x+1[(x \in \mathbf{Z}),]y, x+1[$ $(x \in \mathbf{Z}; x < y < x+1)$, $[y, x+1[(x \in \mathbf{Z}; x < y < x+1)$ and their finite unions, be closed. This space is T_H but not T_{UD}.

Example 3.10. Let the sets $R \backslash \{-1\}, \{0\}$, $\{0, x\}$ $(x \neq -1; x \neq 1)$ and their finite unions, be closed. This space T_I but not T_{UH}.

Example 3.11. Let the sets $]x-1, x]$ $(x \in \mathbf{Z}; x > 0)$, $[x, x+1[(x \in \mathbf{Z}$ $; x < 0)$, $\{x + \frac{1}{2}, y\}$ $(x \in \mathbf{Z}; x < y < x+1)$ and their finite unions, be closed. This space is T_K but not T_J.

4. T_α^A–axioms

In the above classes of topological spaces, T_α, it is easy to prove that their respective characterizations remain equivalent if we change in their statements the arbitrary point x of X for an arbitrary subset A of X. The corresponding classes of spaces obtained in this way will be denoted by T_α^A–spaces. It is clear that $T_\alpha^A \subseteq T_\alpha$ and $T_\alpha \subseteq T_\beta$ implies $T_\alpha^A \subseteq T_\beta^A$.

In the following, we give some new characterizations of $T_{\frac{1}{2}}$, T_D, T_{UD} and T_0–spaces in terms of T_α^A–spaces.

Definition 4.1. A topological space (X, τ) is a $T_{\frac{1}{2}}$–space if for every point x in X, either $\{x\}$ is open or closed.

Remark 4.2. $T_{\frac{1}{2}}$–space were introduced by Levine [4] in terms of the concept of generalized closed sets. Later, they have been studied by McSherry [5] under the name of T_{ES}–spaces; by Jah [3] under the name of T_M–spaces and by Dunham [2].

Proposition 4.3. $T_{\frac{1}{2}} = T^A_{FD} = T^A_F$.

Proof. It is immediate that $T^A_{FD} \subseteq T^A_F$.

If (X,τ) is a $T_{\frac{1}{2}}$-space, then it is T_D [2] and therefore dA is a closed set for every subset A of X. On the other hand, if y is a point in the derived set of a subset A of X, then $\{y\}$ is not open, and so $d\{y\} \neq \emptyset$. Hence (X,τ) is T^A_{FD}.

Let (X,τ) be a T^A_F-space and let x belong to X. If $x \notin dX$, then $\{x\}$ is open, and if $x \in dX$, then $\{x\}$ is closed. Hence (X,τ) is $T_{\frac{1}{2}}$.

Proposition 4.4. $T_D = T^A_D = T^A_{HG} = T^A_{HUD} = T^A_H$.

Proof. It is known that $T_D = T^A_D$ [1].

It is immediate that $T^A_D \subseteq T^A_{HG} \subseteq T^A_{HUD} \subseteq T^A_H$.

Let (X,τ) be a T^A_H-space and let x belong to X. If $x \notin dX$, then $\{x\}$ is open and so $d\{x\}$ is closed, and if $x \in dX$, then $d\{x\}$ is closed. Therefore (X,τ) is T_D.

Proposition 4.5. *A topological space (X,τ) is T_{UD} iff it is T^A_{UH}.*

Proof. It is immediate that $T_{UD} \subseteq T^A_{UH}$.

Let (X,τ) be a T^A_{UH} and let x belong to X. If $x \notin dX$, then $\{x\}$ is open and so $d\{x\}$ is closed, and if $x \in dX$, then $d\{x\}$ is a union of disjoint closed sets. Hence, (X,τ) is T_{UD}.

Proposition 4.6. *A topological space (X,τ) is T_0 iff it is T^A_0.*

Proof. See [6].

Remark 4.7. The only spaces for which $T^A_\alpha = T_\alpha$ are for $\alpha = D$ and $\alpha = 0$. In the other cases, $T^A_\alpha \subseteq T_\alpha$, as it is shown in the following.

In all the examples, X is the set of real numbers and it is understood that the null set and the set X are closed.

Example 4.8. Let the sets $\{x\}(x \geq 0)$, $\{-x,x\}$ $(x > 0)$, $[-x,x]$ $(x > 0)$ and their finite unions, be closed. This space is T_{FD} but it is not $T^A_{FD} = T^A_F$. (Take the subset $[-1,1]$).

4.9. The space of example (3.7) is T_{HG} but it is not $T_D = T^A_{HG} = T^A_{HUD} = T^A_H$.

Example 4.10. Let the sets $]-\infty,0]$, $]0,+\infty[$, $\{x\}$ $(x \neq 0; x \neq 1)$ and their finite unions, be closed. This space is T_G but it is not T^A_I. (Take the subset $]-\infty,0] \cup \{1\}$).

Example 4.11. Let the sets $\{x\}$ $(0 < x \leq 1)$, $\{-x, x\}$ $(0 < x \leq 1)$, $]x, 1]$ $(x \leq -1)$, $[x, 1]$ $(x \leq -1)$, $]0, x[$ $(x > 1)$, $]0, x]$ $(x \geq 1)$ and their finite unions, be closed. This space T_{UD} but it is not T_{UD}^A. (Take the subset $\{0\} \cup [1, 2]$).

4.12. The space of example (3.9) is T_{KUH} but it is not $T_{UD} = T_{UH}^A$.

Example 4.13. Let the sets $\{x\}$ $(x < -1)$ $(0 < x \leq 1)$, $\{-x, x\}$ $(0 < x \leq 1)$, $[-1, 1]$, $[2, 4]$, $]0, x[$ $(1 < x < 2)$ $(4 < x)$, $]0, x]$ $(1 \leq x < 2)(4 \leq x)$, $\{3, x\}$ $(2 \leq x < 4)$ and their finite unions, be closed. This space is T_{KJ} but it is not T_{UI}^A. (Take the subset $\{0\} \cup [1, 4]$).

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