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THE PAINLEVE' PROPERTY, COMPLETE INTERGRABILITY AND BÄCKLUND TRANSFORMATIONS FOR A FAMILY OF LIOUVILLE EQUATIONS

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1. Introduction

For partial differential equations (pde), the painleve' property is defined to mean that the solution of a given pde can be represented locally as a single–valued expansion about its movable singular manifold, [1–3, 10]. That is if $u = u(z_1, \cdots, z_n)$ is a solution of the pde then we can write $u = \phi^{\alpha} \sum_{j=o}^{\infty} u_j \phi^j$, where $\phi = \phi(z_1, \cdots, z_n)$ is analytic function for which $\phi(z_1, \cdots, z_n) = 0$ defines the singular manifold of the pde. The $u_j = u_j(z_1, \cdots, z_n)$ are certain analytic functions of the independent variables $z_1, z_2, \cdots, z_n$ of the pde.

For the painleve' property to be satisfied we require α to be negative integer to make the expansion single–valued. Also we require the expansion u to be self consistent solution to the given pde with the requisite number of arbitrary functions which ϕ itself can be one of them. The powers of ϕ at which these arbitrary functions arise are called resonances.

In fact the painleve' property has shown great importance as its satisfication may provide a simple criterion for complete and partial integrability of pde as well as Bäcklund transformations and Lax pairs, [2,4,11].

The Liouvill's equation has emerged recently in current research in studying the phenomena of non–linear MHD equilibrium and its applications in astrophysics and plasma physics (e.g. flare models and fusion reactor models) [14–17]. However we shall give a detailed analysis in MHD approximation in a forth comming paper [18].

In this paper, we show that a general family of Liouville equations are of Painleve' type and construct a family of Bäcklund transformations of these equations.

2. Liouville Equations and The Painleve' Property

A family of Liouville equations is given by

$$u_{xx} + u_{tt} = ae^{bu} \tag{2.1}$$

where a and b are constant parameters. To apply the painleve' analysis to this family of equations, we first make the following transformation so that we can handle the equations:

$$V = e^{bu} \tag{2.2}$$

Then (2.1) becomes

$$VV_{xx} + VV_{tt} - (V_x)^2 - (V_t)^2 = abV^3 \tag{2.3}$$

Now, let

$$V = \phi^{\alpha} \sum_{j=0}^{\infty} V_j \phi^j \tag{2.4}$$

where $V_j, j = 0, 1, \cdots$ and ϕ are functions of x and t, the independent variables in our case. Then applying (2.4) into (2.3) and by the leading order term analysis, we get $\alpha = -2$ and

$$V_0 = \frac{2}{ab}(\phi_x^2 + \phi_t^2) \tag{2.5}$$

Therefore

$$V = \phi^{-2} \sum_{j=0}^{\infty} V_j \phi^j \tag{2.6}$$

Applying (2.6) in (2.3) we get the recursion relation

$$\begin{aligned} \sum_{m=0}^{j} V_{j-m}(C_m + D_m) &= \sum_{m=0}^{j} (A_{j-m}A_m + B_{j-m}B_m) \\ &\quad + ab \sum_{m=0}^{j} \sum_{k=0}^{m} V_{j-m}V_{m-k}V_k \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} A_j &= V_{j-1,x} + (j-2)V_j\phi_x \\ B_j &= V_{j-1,t} + (j-2)V_j\phi_t \\ C_j &= V_{j-2,xx} + 2(j-3)V_{j-1,x}\phi_x + (j-2)(j-3)V_j\phi_x^2 \\ &\quad +(j-3)V_{j-1}\phi_{xx} \\ D_j &= V_{j-2,tt} + 2(j-3)V_{j-1,t}\phi_t + (j-2)(j-3)V_j\phi_t^2 \\ &\quad +(j-3)V_{j-1}\phi_{tt} \end{aligned} \tag{2.8}$$

Now for different values of j and from eqn. (2.7) we get the following results:

$$\text{For} \quad j=0, \quad V_0 = \frac{2}{ab}(\phi_x^2 + \phi_t^2)$$

Which may be taken in the form

$$V_0 = \frac{2}{ab}(\phi_x \frac{\partial}{\partial x} + \phi_t \frac{\partial}{\partial t})\phi \tag{2.9}$$

For $j = 1$

$$V_1 = \frac{2}{ab}(\phi_x \frac{\partial}{\partial x} + \phi_t \frac{\partial}{\partial t}) \log(\phi_x^2 + \phi_t^2 - \frac{ab}{2}) \tag{2.10}$$

For $j = 2$, we see that V_2 is arbitrary and have the condition $\Psi = 0$, where

$$\begin{aligned} \Psi &= -V_1[-4V_{ox}\phi_x + 2V_1\phi_x^2 - 2V_0\phi_{xx} - 4V_{ot}\phi_t + 2V_1\phi_t^2 \\ &\quad +2V_0\phi_{tt} - 3abV_0] - V_0[V_{0,xx} - 2V_{1,x}\phi_x - V_1\phi_{xx} \\ &\quad +V_{0,tt} - 2V_{1,t}\phi_t - V_1\phi_{tt} + 4\phi_x V_{1,x} + 4V_{1,t}\phi_t \\ &\quad -8V_0\phi_x^2 - 8V_0\phi_t^2] \equiv 0 \end{aligned} \tag{2.11}$$

For $j = 3$, we have

$$\begin{gathered} V_2(C_0 + D_0) + V_1(C_1 + D_1) + V_0(C_2 + D_2) = \\ 2A_0A_1 + 2A_1^2 + 2B_0B_1 + 2B_1^2 + 3ab(V_0^2V_2 + V_0V_1^2) \end{gathered} \tag{2.12}$$

Also for $j = 4$, we have

$$\begin{aligned} V_2(C_2 + D_2) + V_1(C_3 + D_3) + V_0(V_{2,xx} + V_{2,tt}) &= \\ 2V_{2,x}(V_{0,x} - V_1\phi_x) + 2V_{2,t}(V_{0,t} - V_1\phi_t) \\ +V_{1,x}^2 + V_{1,t}^2 + ab(V_0V_2^2 + 3V_1^2V_2) \end{aligned} \tag{2.13}$$

Provided $V_3 = 0, V_4 = 0$. Also for $j = 5$, we are lead to

$$V_1 V2, xx + V_1 V_{2,tt} - 2V_{1,x}V_{2,x} - 2V_{1,t}V_{2,t} = 2abV_1V_2^2 \tag{2.14}$$

With $V_5 = 0$. Lastly for $j = 6$, we get

$$V_2 V_{2,xx} + V_2 V_{2,tt} - V_{2,x}^2 - V_{2,t}^2 = abV_2^3 \tag{2.15}$$

and $V_j = 0$ for $j \geq 3$. Therefore, we get the painleve' expansion

$$V = \phi^{-2}V_0 + \phi^{-1}V_1 + V_2 \tag{2.16}$$

To obtain the resonance, we solve eqn. (2.7) for V_j and get

$$j(j-3)V_jV_0(\phi_x^2 + \phi_t^2) = F(V_{j-1}, V_{j-2}, \cdots, V_0, \phi, \phi_t, \phi_x, \cdots)$$

Therefore the resonance occur for $j = 0$ and $j = 3$. The case for $j = 0$, corresponds to the arbitrary function ϕ itself, and for $j = 3$, since $V_3 = 0$, it is satisfied identically and we are lead to eqn. (2.12) again.

Thus the truncated expansion (2.16), we just have for V proves that the family of Liouville equations satisfy the Painleve' property.

3. Integrability and Bäcklund Transformations

In fact, in Ref. [1,7], it was conjectured that the Painleve' property is sufficient for integrability of the pde addmitting it. However, in Ref.[4], another conjecture was shown and it amounts about the necessity of the painleve' property for the integrability sake and not sufficiency. Therefore the term partial integrability is introduced to denote other additional compatibility conditions that may be added to the satisfication of the painleve' property to ensure complete integrability, [3].

Thus, this general family of Liouville equations under consideration are partially integrable and conditions (2.11)-(2.14) are required for complete integrability. It is worth mentioning that one may use the Schwarzian derivative to simplify these compatibility conditions as in Ref. [2,3] but it is not a straightforward as it contains the Schwarzian derivatives with respect to both x and t.

To look for Bäcklund transformations for this family of equations, we call again equations (2.15) and (2.16), i.e.

$$V = \phi^{-2}V_0 + \phi^{-1}V_1 + V_2$$

and $V_2V_{3,xx} + V_2V_{2,tt} - V_{2,x}^2 - V_{2,t}^2 = abV_2^3$.

We notice that V and V_2 are both solutions of equation (2.3) as expected, [3] and equation (2.16) may be written as

$$\begin{aligned} V &= \frac{2(\phi_x^2+\phi_t^2)}{ab\phi^2} + \frac{4}{ab(\phi_x^2+\phi_t^2-\frac{ab}{2})}[\phi_x^2\phi_{xx} \\ &\quad +2\phi_x\phi_t\phi_{xt} + \phi_t^2\phi_{tt}] + V_2 \end{aligned}$$

Or simply

$$V = \frac{2}{ab\phi}(\phi_x\frac{\partial}{\partial x} + \phi_t\frac{\partial}{\partial t})\log\phi(\phi_x^2+\phi_t^2-\frac{ab}{2}) + V_2 \tag{3.1}$$

Equation (3.1) gives a family of Bäcklund transformations for eqn. (2.3).

Now, with $bu = \ln V$ and $bu_2 = \ln V_2$ eqn. (3.1) becomes

$$e^{bu} = \frac{2}{ab\phi}(\phi_x\frac{\partial}{\partial x} + \phi_t\frac{\partial}{\partial t})\log\phi(\phi_x^2+\phi_t^2-\frac{ab}{2}) + e^{bu}2 \tag{3.2}$$

Where u and u_2 are both solutions of eqns. (2.1) while u_2 and ϕ satisfy the compatibility conditions (2.11)–(2.14).

As a matter of fact, a Bäcklund transformation for the Liouville equation, with $a = 1, b = 2$ was obtained in Ref. [13] by a different approach. Also for the equation $u_{xt} = e^u$ in Ref. [12] by a similar way using the Painleve' analysis.

It is worth mentioning that eqn. (3.2) may give auto–Bäcklund transformations, [11] as well as ordinary Bäcklund transformations as shown for the eqn. $u_{xt} = e^u$ in Ref. [12].

We think that the compatibility conditions, eqns. (2.12)–(2.14), are not just to provide complete integrability but also one may use them to classify some classes of pde as they have some kind of relation with the Bäcklund transformations. This problem in general is a topic for study. Now and will be completed and shown elsewhere.

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