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# MEROMORPHICALLY STARLIKE FUNCTIONS WITH POSITIVE AND FIXED SECOND COEFFICIENTS

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## 1. Introduction

A function $f(z)=\frac{1}{z}+a_0+a_1z+\cdots$ analytic in the annulus $0<|z|<1$ and satisfying the condition

$$\left|\frac{zf'(z)}{f(z)}+1\right|/\left|\frac{zf'(z)}{f(z)}-1\right|<\alpha,\quad 0<\alpha\leq 1$$

for $|z|<1$ is said to be meromorphically starlike of order $\alpha$ and the class of all such functions is denoted by $\Sigma(\alpha)$. The class $\Sigma(1)$ is the same as the class of meromorphically starlike functions in the punctured disk satisfying the condition

$$\operatorname{Re} zf'(z)/f(z)<0\quad \text{for}\quad |z|<1.$$

In [1] K.S. Padmanabhan has obtained the radius of convexity for the class $\Sigma(\alpha)$. Let $\Sigma[\alpha]$ denote the subclass of $\Sigma(\alpha)$ consisting of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}|a_n|z^n. \tag{1}$$

Functions of the form (1) in $\Sigma[\alpha]$ satisfy the coefficient inequality $|a_1|\leq\alpha$[3]. Hence we may take $|a_1|=p\alpha$ where $0\leq p\leq 1$. Let $\sum_p[\alpha]$ be the subclass of $\sum[\alpha]$ consisting of functions of the form $f(z)=\frac{1}{z}+p\alpha z+\sum_{n=2}^{\infty}|a_n|z^n$.

In this paper we obtain coefficient inequalities for the class $\sum_p[\alpha]$ and show that the class $\sum_p[\alpha]$ is closed under *arithmetic mean* and *convex*

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*linear combinations.* Further the radius of convexity is obtained for the class $\sum_p[\alpha]$. While determining the radius of convexity for the class $\sum_p[\alpha]$, we have been able to show that the earlier result of the author [3] follow as a special case of the result derived here. Techniques used are similar to those of H. Silverman [2].

## 2. Coefficient Inequalities

**Theorem 1.** *A function $f(z) = \frac{1}{z} + p\alpha z + \sum_{n=2}^{\infty} |a_n| z^n$ is in the class $\sum_p[\alpha]$ if and only if*

$$\sum_{n=2}^{\infty} |a_n|(n + 1 + n\alpha - \alpha) \leq 2\alpha(1 - p). \tag{2}$$

*Proof.* It is known that $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |a_n| z^n \in \sum[\alpha]$ if and only if $\sum_{n=1}^{\infty} |a_n|(n + 1 + n\alpha - \alpha) \leq 2\alpha$[3].

The result follows by putting $a_1 = p\alpha$ in the above inequality.

**Corollary 1.** *If $f(z) = \frac{1}{z} + p\alpha z + \sum_{n=2}^{\infty} |a_n| z^n$ is in $\sum_p[\alpha]$ then $|a_n| \leq \frac{2\alpha(1-p)}{n+1+n\alpha-\alpha}, n = 2, 3, 4, \cdots$.*

*The estimate is sharp for the function*

$$f(z) = \frac{1}{z} + p\alpha z + \frac{2\alpha(1-p)}{n + 1 + n\alpha - \alpha} z^n.$$

**Corollary 2.** *If $0 \leq p_1 \leq p_2 \leq 1$, then $\sum_{p_2}[\alpha] \subset \sum_{p_1}[\alpha]$.*

## 3. Closure Properties

In this section we shall show that the class $\sum_p[\alpha]$ is closed under *arithmetic mean* and *convex linear combinations.*

**Theorem 2.** *Let $f_i(z) = \frac{1}{z} + p\alpha z + \sum_{n=2}^{\infty} |a_{n_i}| z^n$ for $i = 1, 2, \cdots m$. If $f_i(z) \in \sum_p[\alpha]$ for each $i = 1, 2, \cdots, m$ then the function $g(z) = \frac{1}{z} + p\alpha z + \sum_{n=2}^{\infty} |b_n| z^n$ also is a member of $\sum_p[\alpha]$, where $|b_n| = \frac{1}{m} \sum_{i=1}^{m} |a_{n_i}|$.*

*Proof.* Since $f_i \in \sum_p[\alpha]$ it follows from Theorem 1 that

$$\sum_{n=2}^{\infty} |a_{n_i}|(n + 1 + n\alpha - \alpha) \leq 2\alpha(1 - p)$$

Hence

$$\begin{aligned}\sum_{n=2}^{\infty}|b_n|(n+1+n\alpha-\alpha) &= \sum_{n=2}^{\infty}(\frac{1}{m}\sum_{i=1}^{m}|a_{n_i}|)(n+1+n\alpha-\alpha)\\ &= \frac{1}{m}\sum_{i=1}^{m}\sum_{n=2}^{\infty}|a_{n_i}|(n+1+n\alpha-\alpha)\\ &\leq 2\alpha(1-p)\end{aligned}$$

and the result follows.

**Theorem 3.** *Let $f_1(z)=\frac{1}{z}+p\alpha z$ and $f_n(z)=\frac{1}{z}+p\alpha z+\frac{2\alpha(1-p)}{n+1+n\alpha-\alpha}z^n, n=2,3,\cdots$. Then $f(z)$ is in $\sum_p[\alpha]$ if and only if $f(z)$ can be expressed in the form $f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)$ where $\lambda_n\geq 0$ and $\sum_{n=1}^{\infty}\lambda_n=1$.*

*Proof.* Suppose

$$\begin{aligned}f(z) &= \sum_{n=1}^{\infty}\lambda_n f_n(z)\\ &= \frac{1}{z}+p\alpha z+\sum_{n=2}^{\infty}\frac{\lambda_n(1-p)2\alpha}{n+1+n\alpha-\alpha}z^n.\end{aligned}$$

Since

$$\sum_{n=2}^{\infty}\frac{\lambda_n(1-p)2\alpha}{n+1+n\alpha-\alpha}\frac{n+1+n\alpha-\alpha}{(1-p)2\alpha}=\sum_{n=2}^{\infty}\lambda_n=1-\lambda_1\leq 1,$$

it follows from Theorem 1 that $f(z)\in\sum_p[\alpha]$.

Conversely suppose $f(z)=\frac{1}{z}+p\alpha z+\sum_{n=2}^{\infty}|a_n|z^n$ is in $\sum_p[\alpha]$. From corollary 1, we have

$$|a_n|\leq\frac{2\alpha(1-p)}{n+1+n\alpha-\alpha}.$$

Taking $\lambda_n=\frac{n+1+n\alpha-\alpha}{2\alpha(1-p)}|a_n|, n=2,3,\cdots$. and $\lambda_1=1-\sum_{n=2}^{\infty}\lambda_n$, we get $f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)$.

## 4. Radius of Convexity

**Theorem 4.** *If $f\in\sum_p[\alpha]$ then $f$ is convex in $0<|z|<r=r(\alpha,p)$, where $r(\alpha,p)$ is the largest value for which*

$$3p\alpha r^2+\frac{n(n+2)2\alpha(1-p)}{n+1+n\alpha-\alpha}r^{n+1}\leq 1.(n=2,3,\cdots)$$

*The result is sharp for the function*

$$f_n(z) = \frac{1}{z} + p\alpha z + \frac{2\alpha(1-p)}{n+1+n\alpha-\alpha} z^n \quad \text{for some} \quad n.$$

*Proof.* Since $|zf''(z)/f'(z)+2| < 1$ implies $\mathrm{Re}(1+zf''(z)/f'(z)) < 0$, the function $f(z) = \frac{1}{z} + p\alpha z + \sum_{n=2}^{\infty} |a_n| z^n$ will be convex in any annulus $0 < |z| < r$ for which $|zf''(z)/f'(z)+2| < 1$. But

$$\left|zf''(z)/f'(z)+2\right| \le \frac{2p\alpha r^2 + \sum_{n=2}^{\infty} n(n+1)|a_n| r^{n+1}}{1 - p\alpha r^2 - \sum_{n=2}^{\infty} n|a_n| r^{n+1}} < 1$$

whenever $3p\alpha r^2 + \sum_{n=2}^{\infty} n(n+2)|a_n| r^{n+1} < 1$. Since $f \in \sum_p[\alpha]$, from (2) we may take

$$|a_n| = \frac{2\lambda_n \alpha(1-p)}{n+1+n\alpha-\alpha}, \quad \sum_{n=2}^{\infty} \lambda_n \le 1.$$

For each fixed $r$, choose an integer $n = n(r)$ for which $\frac{n(n+2)r^{n+1}}{n+1+n\alpha-\alpha}$ is maximal. Then

$$\sum_{n=2}^{\infty} n(n+2)|a_n| r^{n+1} \le \frac{n(n+2)2\alpha(1-p)}{n+1+n\alpha-\alpha} r^{n+1}.$$

Now find the value $r_0 = r_0(\alpha, p)$ and the corresponding $n(r_0)$ so that

$$3p\alpha r_0^2 + \frac{n(n+2)2\alpha(1-p)}{n+1+n\alpha-\alpha} r^{n+1} = 1.$$

It is this value $r_0$ for which $f(z)$ is convex in $0 < |z| < r_0$. By taking $p = 1$ and $\alpha = 1$ in Theorem 4 the following result of the author [3] is obtained : If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |a_n| z^n$ is meromorphically starlike then $f(z)$ is convex in $0 < |z| < 1/\sqrt{3}$.

# References

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