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INFINITESIMAL TRANSFORMATIONS WHICH ARE RELATED TO CL–TRANSFORMATION IN SASAKIAN MANIFOLDS (3)

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1. Introduction

A $(2n+1)$–dimensional differentiable manifold M is called to have a Sasakian structure if there is given a positive Riemannian metric g_{ji} and a triplet $(\varphi_k{}^j, \xi^j, \eta_k)$ of $(1,1)$–type tensor field $\varphi_k{}^j$, vector field ξ^j and 1–form η_k in M which satisfy the following equations

$$(1.1)\quad \varphi_j{}^i\varphi_i{}^h = -\delta_j{}^h + \eta_j\xi^h, \quad \varphi_j{}^i\xi^j = 0, \quad \eta_i\varphi_j{}^i = 0, \quad \eta_t\xi^t = 1,$$
$$g_{st}\varphi_j{}^s\varphi_i{}^t = g_{ji} - \eta_j\eta_i, \quad \eta_i = g_{hi}\xi^h$$

and

$$(1.2)\qquad \nabla_i\xi^h = \varphi_i{}^h, \quad \nabla_j\varphi_i{}^h = -g_{ji}\xi^h + \delta_j{}^h\eta_i,$$

where ∇_k indicates the covariant differentiation with respect to g_{ji}. By virtue of the last equation of (1.1) we shall write η^h instead of ξ^h in the sequel. The indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$.

In a Sasakian manifold M, a vector field v^i is called an *infinitesimal CL-transformation* if it satisfies

$$(1.3)\qquad \mathcal{L}_v\left\{ {h \atop j\ i} \right\} = p_j\delta_i{}^h + p_i\delta_j{}^h + \alpha(\eta_j\varphi_i{}^h + \eta_i\varphi_j{}^h)$$

for a certain covector field p_i and a scalar function α where $\mathcal{L}_v$ denotes the Lie derivation with respect to v^i and $\left\{ {h \atop j\ i} \right\}$ is the Riemannian connection.

In a previous paper ([1]), we proved the following

Lemma A. *If a $2n+1(n \geq 1)$ dimensional Sasakian manifold M admits an infinitesimal CL-transformation v^h defined by (1.3), then α is a constant.*

In the present paper, we consider an infinitesimal transformation v^h in a Sasakian manifold M defined by the condition

$$\mathcal{L}_v \left\{ \begin{matrix} h \\ j \;\; i \end{matrix} \right\} = p_j \delta_i^{\ h} + p_i \delta_j^{\ h} + c(\eta_j \alpha_i^{\ h} + \eta_i \alpha_j^{\ h}) \tag{1.4}$$

for a constant c, a certain covector field p_i and a certain $(1,1)$–type tensor field $\alpha_i^{\ h}$ such that

$$g_{hi}\alpha_j^{\ h} = \alpha_{ji} = -\alpha_{ij}. \tag{1.5}$$

The purpose of the present paper is to prove the following theorem.

Main Theorem. *Let a Sasakian manifold of dimension $2n+1(n > 1)$ admits an infinitesimal transformation v^h defined by (1.4). Then $c\alpha_j^{\ t} = c'\varphi_j^{\ t}$, c' being a constant, that is, v^h is an infinitesimal CL-transformation.*

Thus, in a $2n+1(n>1)$ dimensional Sasakian manifold, there exists no infinitesimal transformation v^h satisfying (1.4) and $c\alpha_j^{\ t} \neq c'\varphi_j^{\ t}, c'$ being a constant.

2. Preliminaries

In a $(2n+1)$–dimensional Sasakian manifold M characterized by (1.1) and (1.2), the following identities are well known ([2])

$$K_{kjt}^{\ \ \ h}\eta^t = \delta_k^{\ h}\eta_j - \delta_j^{\ h}\eta_k, \tag{2.1}$$

$$(1) \quad \eta^k K_{kji}^{\ \ \ t} = \eta^t g_{ji} - \eta_i \delta_j^{\ t}, \quad (2) \quad K_{kji}^{\ \ \ t}\eta_t = \eta_k g_{ji} - \eta_j g_{ki}, \tag{2.2}$$

$$K_{jt}\eta^t = 2n\eta_j, \tag{2.3}$$

$$\varphi^{ts}K_{tsj}^{\ \ \ h} + 2\varphi_j^{\ t}K_t^{\ h} = 2(2n-1)\varphi_j^{\ h}, \tag{2.4}$$

where $K_{kji}^{\ \ \ h}$ and K_{ji} are the curvature tensor and the Ricci tensor of M respectively.

Let us recall the definition of Lie derivation. For any vector field v^h, we have the following identities

$$\eta^j\eta^i\mathcal{L}_v g_{ji} = 2\eta^i\mathcal{L}_v\eta_i, \tag{2.5}$$

$$\nabla_k \mathcal{L}_v g_{ji} = g_{ti}\mathcal{L}_v\left\{ {t \atop k\ j} \right\} + g_{jt}\mathcal{L}_v\left\{ {t \atop k\ i} \right\} \tag{2.6}$$

$$\mathcal{L}_v\left\{ {h \atop j\ i} \right\} = \nabla_j\nabla_i v^h + v^t K_{tji}{}^h \tag{2.7}$$

and

$$\mathcal{L}_v K_{kji}{}^h = \nabla_k\mathcal{L}_v\left\{ {h \atop j\ i} \right\} - \nabla_j\mathcal{L}_v\left\{ {h \atop k\ i} \right\} \tag{2.8}$$

3. Infinitesimal transformation defined by (1.4)

First of all, substituting (1.4) into (2.8), we obtain

$$\begin{aligned}\mathcal{L}_v K_{kji}{}^h \ =\ & (\nabla_k p_j - \nabla_j p_k)\delta_i{}^h + (\nabla_k p_i)\delta_j{}^h - (\nabla_j p_i)\delta_k{}^h \\ & + c[2\varphi_{kj}\alpha_i{}^h + \varphi_{ki}\alpha_j{}^h - \varphi_{ji}\alpha_k{}^h \\ & + \eta_j\nabla_k\alpha_i{}^h - \eta_k\nabla_j\alpha_i{}^h + \eta_i(\nabla_k\alpha_j{}^h - \nabla_j\alpha_k{}^h)],\end{aligned}$$

from which, transvecting with η_h,

$$\begin{aligned}\eta_h\mathcal{L}_v K_{kji}{}^h \ =\ & (\nabla_k p_j - \nabla_j p_k)\eta_i + (\nabla_k p_i)\eta_j - (\nabla_j p_i)\eta_k \\ & + c[2\varphi_{kj}\eta_t\alpha_i{}^t + \varphi_{ki}\eta_t\alpha_j{}^t - \varphi_{ji}\eta_t\alpha_k{}^t \\ & + \eta_t(\eta_j\nabla_k\alpha_i{}^t - \eta_k\nabla_j\alpha_i{}^t) + \eta_i\eta_t(\nabla_k\alpha_j{}^t - \nabla_j\alpha_k{}^t)].\end{aligned} \tag{3.1}$$

Taking the Lie derivative of the both sides of (2) of (2.2), we obtain

$$\eta_t\mathcal{L}_v K_{kji}{}^t + K_{kji}{}^t\mathcal{L}_v\eta_t = \eta_k\mathcal{L}_v g_{ji} - \eta_j\mathcal{L}_v g_{ki} + g_{ji}\mathcal{L}_v\eta_k - g_{ki}\mathcal{L}_v\eta_j, \tag{3.2}$$

from which, transvecting with η^k and taking account of (1.5),

$$\begin{aligned}\mathcal{L}_v g_{ji} \ =\ & \eta^t(\nabla_t p_j - \nabla_j p_t)\eta_i + \eta_j\eta^t\nabla_t p_i - \nabla_j p_i \\ & + c\eta_t[\eta_j\eta^k\nabla_k\alpha_i{}^t + \eta_i\eta^k\nabla_k\alpha_j{}^t - \nabla_j\alpha_i{}^t] + \eta_j\eta^t\mathcal{L}_v g_{ti}.\end{aligned} \tag{3.3}$$

Taking account of the symmetry of (3.3), we obtain

$$\begin{aligned}&\eta^t(\eta_j\nabla_i p_t - \eta_i\nabla_j p_t) - (\nabla_j p_i - \nabla_i p_j) - c\eta_t(\nabla_j\alpha_i{}^t - \nabla_i\alpha_j{}^t) \\ &\qquad + \eta^t(\eta_j\mathcal{L}_v g_{ti} - \eta_i\mathcal{L}_v g_{tj}) = 0,\end{aligned}$$

from which, transvecting with η^i,

$$(3.4)\qquad \eta^t\mathcal{L}_v g_{tj} = (2\tau+\mu)\eta_j - \eta^t(2\nabla_j p_t - \nabla_t p_j) + c\eta_t\eta^s\nabla_s\alpha_j{}^t,$$

where we have put

$$\eta^k\eta^i\nabla_k p_i = \mu, \qquad \eta^j\eta^i\mathcal{L}_v g_{ji} = 2\tau.$$

Substituting (3.4) into (3.3), we obtain

$$\begin{aligned}(3.5)\ \mathcal{L}_v g_{ji} \;=\;& \eta^t\eta_i(\nabla_t p_j - \nabla_j p_t) - \nabla_j p_i - c\eta_t(\nabla_j\alpha_i{}^t - \eta_i\eta^s\nabla_s\alpha_j{}^t)\\ &+\eta_j[(2\tau+\mu)\eta_i + 2\eta^t(\nabla_t p_i - \nabla_i p_t) + 2c\eta_t\eta^s\nabla_s\alpha_i{}^t].\end{aligned}$$

Taking account of the symmetry of (3.5), we obtain

$$\begin{aligned}(3.6)\qquad &\nabla_j p_i - \nabla_i p_j - \eta^t[\eta_j(\nabla_t p_i - \nabla_i p_t) - \eta_i(\nabla_t p_j - \nabla_j p_t)]\\ &+c\eta_t[\nabla_j\alpha_i{}^t - \nabla_i\alpha_j{}^t - \eta^s(\eta_j\nabla_s\alpha_i{}^t - \eta_i\nabla_s\alpha_j{}^t)] = 0.\end{aligned}$$

On the other hand, substituting (3.1) into (3.2), we obtain

$$\begin{aligned}K_{kji}{}^t\mathcal{L}_v\eta_t \;=\;& -(\nabla_k p_j - \nabla_j p_k)\eta_i - (\nabla_k p_i)\eta_j + (\nabla_j p_i)\eta_k\\ &-c[2\varphi_{kj}\eta_t\alpha_i{}^t + \varphi_{ki}\eta_t\alpha_j{}^t - \varphi_{ji}\eta_t\alpha_k{}^t\\ (3.7)\qquad &+\eta_t(\eta_j\nabla_k\alpha_i{}^t - \eta_k\nabla_j\alpha_i{}^t) + \eta_i\eta_t(\nabla_k\alpha_j{}^t - \nabla_j\alpha_k{}^t)]\\ &+g_{ji}\mathcal{L}_v\eta_k + \eta_k\mathcal{L}_v g_{ji} - g_{ki}\mathcal{L}_v\eta_j - \eta_j\mathcal{L}_v g_{ki},\end{aligned}$$

from which, transvecting with φ^{kj},

$$\begin{aligned}\varphi^{kj}K_{kji}{}^t\mathcal{L}_v\eta_t =\;& [\varphi^{kj}\{-(\nabla_k p_j - \nabla_j p_k) - c\eta_t(\nabla_k\alpha_j{}^t - \nabla_j\alpha_k{}^t)\}]\eta_i\\ (3.8)\qquad & -2(2n+1)c\eta_t\alpha_i{}^t - 2\varphi_i{}^t\mathcal{L}_v\eta_t.\end{aligned}$$

Transvecting (3.8) with η^i and taking account of (2.1), we obtain

$$(3.9)\qquad 0 = \varphi^{kj}\{\nabla_j p_k - \nabla_k p_j + c\eta_t(\nabla_k\alpha_j{}^t - \nabla_j\alpha_k{}^t)\}.$$

Substituting (3.9) into (3.8), we obtain

$$(3.10)\qquad \varphi^{kj}K_{kji}{}^t\mathcal{L}_v\eta_t = -2(2n+1)c\eta_t\alpha_i{}^t - 2\varphi_i{}^t\mathcal{L}_v\eta_t.$$

Transvecting (3.7) with g^{ji} and substituting (3.4) into it, we obtain

$$\begin{aligned}(3.11)\qquad K_k{}^t\mathcal{L}_v\eta_t \;=\;& -(2\tau+\mu)\eta_k - 3c\varphi_k{}^s\eta_t\alpha_s{}^t + 2n\mathcal{L}_v\eta_k\\ &+(\nabla_t p^t - c\eta^t\nabla_s\alpha_t{}^s + g^{ji}\mathcal{L}_v g_{ji})\eta_k,\end{aligned}$$

from which, transvecting with η^k and taking account of (2.3),

$$0 = -(2\tau + \mu) + \nabla_t p^t - c\eta^t \nabla_s \alpha_t{}^s + g^{ji}\mathcal{L}_v g_{ji}. \tag{3.12}$$

Substituting (3.12) into (3.11), we obtain

$$K_k{}^t \mathcal{L}_v \eta_t = -3c\varphi_k{}^s \eta_t \alpha_s{}^t + 2n\mathcal{L}_v \eta_k. \tag{3.13}$$

Substituting (3.10) and (3.13) into the equation

$$\varphi^{kj} K_{kji}{}^t \mathcal{L}_v \eta_t + 2\varphi_i{}^k K_k{}^t \mathcal{L}_v \eta_t = 2(2n-1)\varphi_i{}^t \mathcal{L}_v \eta_t,$$

which is obtained from (2.4), we obtain

$$(n-1)\eta_t \alpha_i{}^t = 0.$$

Thus we have the following

Theorem. *If a $2n+1(n > 1)$ dimensional Sasakian manifold admits an infinitesimal transformation defined by (1.4), then the relation*

$$\eta_t \alpha_j{}^t = 0 \tag{3.14}$$

holds good.

4. Proof of the main theorem

The condition (1.4) is rewritten as

$$\nabla_j \nabla_i v^h + v^t K_{tji}{}^h = p_j \delta_i^h + p_i \delta_j^h + c(\eta_j \alpha_i{}^h + \eta_i \alpha_j{}^h). \tag{4.1}$$

Contracting with respect to h and i in (4.1) and taking account of (3.14), we obtain

$$\nabla_j(\nabla_t v^t) = 2(n+1)p_j.$$

Thus, p_i is a gradient vector field, that is, the following equation holds good:

$$\nabla_j p_i - \nabla_i p_j = 0. \tag{4.2}$$

Differentiating (3.14) covariantly, we obtain

$$\eta_t \nabla_k \alpha_j{}^t = -\varphi_{kt} \alpha_j{}^t, \tag{4.3}$$

and consequently substituting (4.2) and (4.3) into (3.6), we find

$$\varphi_{jt} \alpha_i{}^t - \varphi_{it} \alpha_j{}^t = 0. \tag{4.4}$$

On the other hand, substituting (4.2) and (4.3) into (3.5), we obtain

$$\mathcal{L}_v g_{ji} = -\nabla_j p_i + c\varphi_{jt}\alpha_i{}^t + (2\tau+\mu)\eta_j\eta_i. \tag{4.5}$$

Substituting (1.4) and (4.5) into (2.6) and taking account of (3.14), we obtain

$$\begin{aligned} -\nabla_k\nabla_j p_i + c(\eta_j\alpha_{ik} + \varphi_{jt}\nabla_k\alpha_i{}^t) + \eta_j\eta_i\nabla_k(2\tau+\mu) \\ +(2\tau+\mu)(\varphi_{kj}\eta_i + \varphi_{ki}\eta_j) \\ = 2p_k g_{ji} + p_j g_{ki} + p_i g_{jk} + c(\eta_j\alpha_{ki} + \eta_i\alpha_{kj}). \end{aligned} \tag{4.6}$$

Substituting (4.6) into the Ricci identity:

$$\nabla_k\nabla_j p_i - \nabla_j\nabla_k p_i = -K_{kji}{}^t p_t,$$

transvecting it with η^k and taking account of (1) of (2.2), we obtain

$$\begin{aligned} c\varphi_{jt}\eta^k\nabla_k\alpha_i{}^t + \eta^k\eta_j\eta_i\nabla_k(2\tau+\mu) - \eta_i\nabla_j(2\tau+\mu) \\ -(2\tau+\mu)\varphi_{ji} + 2c\alpha_{ji} = 0, \end{aligned}$$

from which, transvecting with $\varphi_h{}^j$,

$$\begin{aligned} c\eta^k\nabla_k\alpha_{ih} + \eta_i\varphi_h{}^k\nabla_k(2\tau+\mu) - (2\tau+\mu)(g_{hi} - \eta_h\eta_i) \\ -2c\varphi_h{}^k\alpha_{ki} = 0. \end{aligned} \tag{4.7}$$

Transvecting (4.7) with η^i and using the fact that

$$\eta^k\eta^i\nabla_k\alpha_{ih} = \eta^k\nabla_k(\eta^i\alpha_{ih}) = 0,$$

we obtain

$$\varphi_h{}^t\nabla_t(2\tau+\mu) = 0,$$

from which, transvecting with $\varphi_k{}^h$,

$$\nabla_k(2\tau+\mu) = \eta_k\eta^t\nabla_t(2\tau+\mu). \tag{4.8}$$

Applying $\varphi^{jk}\nabla_j$ to (4.8), we can easily see that

$$\eta^t\nabla_t(2\tau+\mu) = 0$$

and hence

$$2\tau+\mu = \text{constant}.$$

Substituting (4.8) into (4.7), we obtain

$$c\eta^k \nabla_k \alpha_{ih} - (2\tau + \mu)(g_{hi} - \eta_h \eta_i) - 2c\varphi_h{}^t \alpha_{ti} = 0. \tag{4.9}$$

Taking the symmetric part of (4.9) and taking account of (4.4), we obtain

$$(2\tau + \mu)(g_{hi} - \eta_h \eta_i) - 2c\varphi_{ht} \alpha_i{}^t = 0,$$

from which, transvecting with $\varphi_k{}^h$ and taking account of (3.14),

$$2c\alpha_{ki} = (2\tau + \mu)\varphi_{ki}.$$

Taking account of the fact that $2\tau + \mu$ is a constant, the proof of the main theorem is completed.

References

[1] S.-S. Eum, *Infinitesimal transformations which are related to CL-transformation in Sasakian manifolds*, Sung Kyun Kwan Univ. Joun., 37-1 (1986), 69-80.

[2] K. Yano, *On contact conformal connections*, Kōdai Math. Sem. Rep., 28(1976), 90-103.

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