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ISOMETRIES OF $AlgL_n$

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1. Introduction

The study of non–self–adjoint operator algebras on Hilbert space was only beginned by W.B. Arveson (1) in 1974. Recently, such algebras have been found to be of use in physics, in electrical engineering, and in general systems theory. Of particular interest to mathematicians are reflexive algebras with commutative lattices of invariant subspaces. The algebras $AlgL_n$ are important classes of such algebras. These algebras possess many surprising properties related to isometries, isomorphisms, cohomology and extreme points. In this paper, we shall investigate the isometric maps of these algebras.

First we will introduce the terminologies which are used in this paper. Let $\mathbf{H}$ be a complex Hilbert space and let $\mathcal{A}$ be a subset of $\mathbf{B}(\mathbf{H})$, the class of all bounded operators acting on $\mathbf{H}$. If $\mathcal{A}$ is a vector space over $\mathbf{C}$ and if $\mathcal{A}$ is closed under the composition of maps, then $\mathcal{A}$ is called an algebra. $\mathcal{A}$ is called a self-adjoint algebra provided A^* is in $\mathcal{A}$ for every A in $\mathcal{A}$. Otherwise $\mathcal{A}$ is called a non-self-adjoint algebra. A linear map φ of one algebra $\mathcal{A}_1$ into another algebra $\mathcal{A}_2$ is an isometry if it preserves norm. If $\mathcal{L}$ is a lattice of orthogonal projections acting on $\mathbf{H}$, $Alg\mathcal{L}$ denotes the algebra of all bounded operators acting on $\mathbf{H}$ that leave invariant every orthogonal projection in $\mathcal{L}$. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on $\mathbf{H}$, containing O and I. Dually, if $\mathcal{A}$ is a subalgebra of $\mathbf{B}(\mathbf{H})$, then $Lat\mathcal{A}$ is the lattice of all orthogonal projections invariant for each operator in $\mathcal{A}$. An algebra $\mathcal{A}$ is reflexive if $\mathcal{A} = AlgLat\mathcal{A}$ and a lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = LatAlg\mathcal{L}$. A lattice $\mathcal{L}$ is a commutative subspace lattice, or CSL, if each pair of projections in $\mathcal{L}$ commutes; $Alg\mathcal{L}$ is then called a CSL-algebra.

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If $x_1, x_2, \cdots, x_n$ are vectors in some Hilbert space, then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2 \cdots, x_n$.

2. Isometries of $AlgL_{2n}$ and $AlgL_{2n+1}$

Let L_{2n} (or L_{2n+1}) be the subspace lattice of orthogonal projections generated by $\{[e_1], [e_3], \cdots, [e_{2n-1}], [e_1, e_2, e_3], [e_3, e_4, e_5], \cdots, [e_{2n-3}, e_{2n-2}, e_{2n-1}], [e_{2n-1}, e_{2n}]\}$ (or $\{[e_1], [e_{2i+1}], [e_{2i-1}, e_{2i}, e_{2i+1}] : i = 1, 2, \cdots, n\}$) and let $\mathcal{A}_{2n}$ (or $\mathcal{A}_{2n+1}$) be the algebra considering of all bounded operators acting on $2n$(or $2n+1$)–dimensional complex Hilbert space $\mathbf{H}$ of the form

$$\begin{bmatrix} * & * & & & & & \\ & * & & & & & \\ & * & * & * & & & \\ & & & * & & & \\ & & & * & \cdot & & \\ & & & & & \cdot & \\ & & & & & \cdot & * \\ & & & & & & * \end{bmatrix} \quad (\text{ or } \quad \begin{bmatrix} * & * & & & & & \\ & * & & & & & \\ & * & * & * & & & \\ & & & * & & & \\ & & & * & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot & \\ & & & & & & * & * \end{bmatrix}),$$

where all non–starred entries are zero, with an orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$ (or $\{e_1, e_2, \cdots, e_{2n+1}\}$). Then the lattices L_{2n} and L_{2n+1} are commutative and reflexive. The algebras $\mathcal{A}_{2n}$ and $\mathcal{A}_{2n+1}$ are reflexive. Also we have $AlgL_{2n} = \mathcal{A}_{2n}$ and $AlgL_{2n+1} = \mathcal{A}_{2n+1}(9)$.

Let i and j be positive integers. Then E_{ij} is the matrix whose (i,j)–component is 1 and all other components are zero. Let x and y be two nonzero vectors in a Hilbert space $\mathbf{H}$. Then $x^* \otimes y$ is a rank one operator defined by $x^* \otimes y(h) = \langle h, x\rangle y$ for all h in $\mathbf{H}$.

Lemma 2.1 [12]. *Let $\mathcal{L}$ be a commutative lattice and let x and y be two vectors. Then $x^* \otimes y$ is in $Alg\mathcal{L}$ if and only if there exists E in $\mathcal{L}$ such that y in E and x in $E_-^\perp$, where*

$$E_- = \vee\{F : F \text{ is in } \mathcal{L} \text{ and } F \not\geq E\} \quad \text{and } E_-^\perp = (E_-)^\perp.$$

Let $\varphi : AlgL_{2n} \to AlgL_{2n}$ (or $AlgL_{2n+1} \to AlgL_{2n+1}$) be a surjective isometry. Then we have the following lemmas.

Lemma 2.2 [13]. *Let I be the identity operator, let $\varphi(I) = A$, and let E be a projection in L_{2n} whose rank is at least two. Then $\|Ax\| = \|x\|$ for all x in $E_-^\perp$.*

Lemma 2.3 [13]. *If $\varphi(I) = A$ and if $x^* \otimes x$ is $AlgL_{2n}$, then $\|Ax\| = \|x\|$.*

Theorem 2.4. *If $\varphi : AlgL_{2n} \to AlgL_{2n}$ (or $AlgL_{2n+1} \to AlgL_{2n+1}$) is a surjective isometry, then $\varphi(I)$ is a unitary diagonal operator in $AlgL_{2n}$ (or $AlgL_{2n+1}$).*

Proof. Let $\varphi(I) = A = (a_{ij})$ be in $AlgL_{2n}$. Since $e_i^* \otimes e_i$ is in $AlgL_{2n}$ for each $i = 1, 2, \cdots, 2n$, $|a_{ii}| = 1$ for each odd number i. Since $\|A\| = \|I\| = 1$, we have

$$a_{12} = 0, a_{32} = a_{34} = 0, \cdots, a_{2n-1,2n-2} = a_{2n-1,2n} = 0.$$

Hence $\varphi(I) = A$ is a diagonal matrix and $|a_{ii}| = 1$ for each $i = 1, 2, \cdots, 2n$ (or $2n + 1$). So $A = \varphi(I)$ is a unitary diagonal operator in $AlgL_{2n}$ (or $AlgL_{2n+1}$).

Let $\varphi(I) = U$. Then UA and U^*A are in $AlgL_{2n}$ (or $AlgL_{2n+1}$) for all A in $AlgL_{2n}$ (or $AlgL_{2n+1}$). Define $\widehat{\varphi} : Alg_{2n} \to AlgL_{2n}$ (or $AlgL_{2n+1} \to AlgL_{2n+1}$) by $\widehat{\varphi}(A) = U^*\varphi(A)$ for every A in $AlgL_{2n}$ (or $AlgL_{2n+1}$). Then $\widehat{\varphi}$ is a surjective isometry and $\widehat{\varphi}(I) = I$. Let $\mathcal{M}$ be the smallest von Neumann algebra containing L_{2n} (or L_{2n+1}). Then $\mathcal{M} = (AlgL_{2n}) \cap (AlgL_{2n})^*$ (or $\mathcal{M} = (AlgL_{2n+1}) \cap (AlgL_{2n+1})^*$), where $(Alg\mathcal{L})^* = \{A^* : A \text{ is in } Alg\mathcal{L}\}$ for any subspace lattice $\mathcal{L}$.

Lemma 2.5 [11]. *Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be C^*-algebras and let $\varphi : \mathcal{U}_1 \to \mathcal{U}_2$ be a linear map which carries the identity in $\mathcal{U}_1$ into the identity in $\mathcal{U}_2$ and $\|\varphi(A)\| = \|A\|$ for all normal operators A in $\mathcal{U}_1$. Then φ preserves adjoints, i.e, $\varphi(A^*) = \varphi(A))^*$ for all A in $\mathcal{U}_1$.*

Definition 2.6 [9]. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be C^*-algebras. A Jordan isomorphism or C^*-isomorphism $\varphi : \mathcal{U}_1 \to \mathcal{U}_2$ is a bijective linear map such that $\varphi(A^n) = (\varphi(A))^n$ for all A in $\mathcal{U}_1$ and $\varphi(A) = (\varphi(A))^*$ whenever A is self-adjoint in $\mathcal{U}_1$.

Lemma 2.7 [11]. *(1) A linear bijection φ of one C^*-algebra $\mathcal{U}_1$ onto another $\mathcal{U}_2$ which is isometric is a C^*-isomorphism followed by left multiplication by a fixed unitary operator, viz, $\varphi(I)$.*

(2) A C^*-isomorphism φ of a C^*-algebra $\mathcal{U}_1$ onto a C^*-algebra $\mathcal{U}_2$ is isometric and preserves commutativity.

Lemma 2.8. *Let $\widehat{\varphi} : AlgL_{2n} \to AlgL_{2n}$ (or $AlgL_{2n+1} \to AlgL_{2n+1}$) be a surjective isometry defined by $\widehat{\varphi}(A) = U^*\varphi(A)$ for all A in $AlgL_{2n}$ (or $AlgL_{2n+1}$), where $U = \varphi(I)$. Then $\widehat{\varphi}(\mathcal{M}) = \mathcal{M}$.*

Proof. Since $\mathcal{M}$ is C^*–algebra, $\widehat{\varphi}(I) = I$ and $\widehat{\varphi}$ is an isometry, we have $\varphi|_{\mathcal{M}}$preserves adjoints by Lemma 2.5. If $\widehat{\varphi}(A)$ is in $\widehat{\varphi}(\mathcal{M})$, then A is in $\mathcal{M}$ and so A^* is in $\mathcal{M}$. Hence $\widehat{\varphi}(A^*) = (\widehat{\varphi}(A))^*$ is in $\widehat{\varphi}(\mathcal{M})$ and hence $\widehat{\varphi}(\mathcal{M})$ is self–adjoint. Thus $\widehat{\varphi}(\mathcal{M}) \subset \mathcal{M}$. Since $\widehat{\varphi}|_{\mathcal{M}}: \mathcal{M} \to \mathcal{M}$ is an injective linear map and $\mathcal{M}$ is a finite dimensional vector space, we have $\widehat{\varphi}|_{\mathcal{M}}: \mathcal{M} \to \mathcal{M}$ is onto. Hence $\widehat{\varphi}(\mathcal{M}) = \mathcal{M}$.

Corollary 2.9. *If $\varphi : AlgL_{2n} \to AlgL_{2n}$ (or $AlgL_{2n+1} \to AlgL_{2n+1}$) is a surjective isometry such that $\varphi(I) = I$, then $\varphi(\mathcal{M}) = \mathcal{M}$.*

Lemma 2.10. *Let $\varphi : AlgL_{2n} \to AlgL_{2n}$ (or $AlgL_{2n+1} \to AlgL_{2n+1}$) be a surjective isometry such that $\varphi(I) = I$, Then E is a projection in $\mathcal{M}$ if and only if $\varphi(E)$ is a projection in $\mathcal{M}$.*

Proof. Suppose that E is a projection in $\mathcal{M}$. Since $\varphi|_{\mathcal{M}}$ is a Jordan isomorphism, $\varphi(E) = \varphi(E^*) = \varphi(E)^*$ and $\varphi(E) = \varphi(E^2) = \varphi(E)^2$. So $\varphi(E)$ is a projection in $\mathcal{M}$ because $\varphi(\mathcal{M}) = \mathcal{M}$. Conversely, suppose that $\varphi(E)$ is a projection in $\mathcal{M}$. Then since $\varphi^{-1}|_{\mathcal{M}}$is a Jordan isomorphism, $\varphi^{-1} \circ \varphi(E) = E$ is a projection in $\mathcal{M}$.

Lemma 2.11 [11]. *Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be C^*-algebras and $\varphi : \mathcal{U}_1 \to \mathcal{U}_2$ a C^*-isomorphism. Then $\varphi(BAB) = \varphi(B)\varphi(A)\varphi(B)$ for all A, B in $\mathcal{U}_1$.*

Let E and F be orthogonal projections acting on a Hilbert space $\mathbf{H}$. Then the partial order relation $\leq$ is described as follows ;

$$E \leq F \quad \text{if and only if} \quad EF = FE = E.$$

Theorem 2.12. *Let $\varphi : AlgL_{2n} \to AlgL_{2n}$ (or $AlgL_{2n+1} \to AlgL_{2n+1}$) be a surjective isometry such that $\varphi(I) = I$. Then $\varphi([e_i])$ is a rank one operator for all $i = 1, 2, \cdots, 2n$ (or $i = 1, 2, \cdots, 2n+1$).*

Proof. For given $k = 1, 2, \cdots, 2n$ (or $k = 1, 2, \cdots, 2n+1$), $[e_k]$ is a projection in $\mathcal{M}$. By Lemma 2.10, $\varphi([e_k])$ is a projection in $\mathcal{M}$. Let E be a non–zero projection in $\mathcal{M}$ such that $E \leq \varphi([e_k])$. Then there exists F in $\mathcal{M}$ such that $\varphi(F) = E$ and F is a projection by Lemma 2.10. Since $F[e_k] = [e_k]F$ and $\varphi|_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is an isometry such that $\varphi(I) = I$, it follows by Lemma 2.7, $\varphi(F)\varphi([e_k]) = \varphi([e_k])\varphi(F)$. Since $F[e_k] = [e_k]F[e_k]$, $\varphi(F[e_k]) = \varphi([e_k]F[e_k]) = \varphi([e_k])\varphi(F)\varphi([e_k])$. Hence $\varphi(F[e_k]) = \varphi(F)\varphi([e_k])$. Since $E\varphi([e_k]) = E$, we have

$$\varphi(F) = E = E\varphi([e_k]) = \varphi(F)\varphi([e_k]) = \varphi(F[e_k]).$$

So $F = F[e_k]$ because φ is an injection. Thus $F \leq [e_k]$. Since $\|E\| = \|\varphi(F)\| = \|F\| \neq 0$. $F = [e_k]$. So $E = \varphi(F) = \varphi([e_k])$. that is, $\varphi([e_k])$ is a minimal projection in $\mathcal{M}$. Thus $\varphi([e_k])$ is a rank one operator for all $k = 1, 2, \cdots, 2n$ (resp. $k = 1, 2, \cdots, 2n+1$).

Lemma 2.13 [8]. *Let R be an operator and suppose that there is a non-negative number M and a positive number N such that $\|R + \alpha I\|^2 \leq M^2 + |\alpha|^2$ for all α in $\mathbf{C}$ with $|\alpha| \geq N$. Then $R = 0$.*

Lemma 2.14 [13]. *Let $\varphi : AlgL_{2n} \to AlgL_{2n}$ (or $AlgL_{2n+1} \to AlgL_{2n+1}$) be a surjective isometry such that $\varphi(I) = I$, and let P be a projection in $\mathcal{M}$ and let T be in $AlgL_{2n}$ (or $AlgL_{2n+1}$) with $T = PTP^{\perp}$. Then $\varphi(T) = \varphi(P)\varphi(T)\varphi(P)^{\perp} + \varphi(P)^{\perp}\varphi(T)\varphi(P)$.*

Theorem 2.15 [8]. *Let $\varphi : AlgL_2 \to AlgL_2$ be a surjective isometry such that $\varphi(I) = I$ and $\varphi(E_{ii}) = E_{ii}$ for $i = 1, 2$. Then there exists a unitary operator U such that $\varphi(I) = UAU^*$ for every A in $AlgL_2$.*

Theorem 2.16. *Let $\varphi : AlgL_{2n} \to AlgL_{2n}$ be a surjective isometry such that $\varphi(I) = I$ and $\varphi(E_{ii}) = E_{kk}$, $\varphi(E_{jj}) = E_{mm}$ for $i, j = 1, 2, \cdots, 2n$. If $|i - j| = 1$, then $|k - m| = 1$.*

Proof. Let $i = 2r-1$ and $j = 2r$ for $r = 1, 2, \cdots, n$. Then $E^{\perp}_{2r,2r}E_{2r-1,2r}E_{2r,2r} = E_{2r-1,2r}$ and

$$E_{2r-1,2r-1}E_{2r-1,2r}E^{\perp}_{2r-1,2r-1} = E_{2r-1,2r}.$$

From Lemma 2.14

$$\varphi(E_{2r-1,2r}) = E^{\perp}_{mm}\varphi(E_{2r-1,2r})E_{mm} + E_{mm}\varphi(E_{2r-1,2r})E^{\perp}_{mm} \quad \text{and}$$

$$(*) \quad \varphi(E_{2r-1,2r}) = E_{kk}\varphi(E_{2r-1,2r})E^{\perp}_{kk} + E^{\perp}_{kk}\varphi(E_{2r-1,2r})E_{kk}$$

So we can get the following from the second equation of $(*)$;

(1) If k is 1, then $\varphi(E_{2r-1,2r})$ is a matrix all of whose entries are zero except for the $(1, 2)$–component.

(2) If k is an odd number and $k \neq 1$, then $\varphi(E_{2r-1,2r})$ is a matrix all of whose entries are zero except for the $(k, k-1)$–component and the $(k, k+1)$–component.

(3) If k is an even number and $k \neq 2n$, then $\varphi(E_{2r-1,2r})$ is a matrix all of whose entries are zero except for the $(k-1, k)$–component and the $(k+1, k)$–component.

(4) If k is $2n$, then $\varphi(E_{2r-1,2r})$ is a matrix all of whose entries are zero except for the $(2n-1, 2n)$–component.

From the first equation of $(*)$ we have the following;

(a) If m is 1, then $\varphi(E_{2r-1,2r})$ is a matrix all of whose entries are zero except for the $(1, 2)$–component.

(b) If m is an odd number and $m \neq 1$, then $\varphi(E_{2r-1,2r})$ is a matrix all of whose entries are zero except for the $(m, m-1)$–component and the $(m, m+1)$–component.

(c) If m is an even number and $m \neq 2n$, then $\varphi(E_{2r-1,2r})$ is a matrix all of whose entries are zero except for the $(m-1, m)$–component and the $(m+1, m)$–component.

(d) If m is $2n$, then $\varphi(E_{2r-1,2r})$ is a matrix all of whose entries are zero except for the $(2n-1, 2n)$–component

Then the following cannot happen at the same time:
(1) and (a) because $k \neq m$.
(1) and (b) because $k = 1$ and $m \geq 3$.
(1) and (d) because $k = 1$ and $m = 2n (n > 1)$
(2) and (a) because $k \geq 3$ and $m = 1$.
(2) and (b) because $k \neq m$.
(3) and (c) because $k \neq m$.
(3) and (d) because $k \leq 2(n-1)$ and $m = 2n$.
(4) and (c) because $k = 2n$ and $m \leq 2(n-1)$.
(4) and (d) because $k \neq m$.
Then the following can happen at the same time;

(1) and (c) if $k = 1$ and $m = 2$. In this case, $\varphi(E_{2r-1,2r})$ is a matrix all of whose entries are zero except for the $(1, 2)$–component.

(2) and (c) if $|k - m| = 1$. In this case, $\varphi(E_{2r-1,2r})$ is a matrix all of whose entries are zero except for either the $(k, k-1)$ component or the $(k, k+1)$–component.

(2) and (d) if $k = 2n-1$ and $m = 2n$. In this case, $\varphi(E_{2r-1,2r})$ is a matrix all of whose entries are zero except for the $(2n-1, 2n)$–component.

(3) and (a) if $k = 2$ and $m = 1$. In this case, $\varphi(E_{2r-1,2r})$ is a matrix all of whose entries are zero except for the $(1, 2)$–component.

(3) and (b) if $|k-m|=1$. In this case, $\varphi(E_{2r-1,2r})$ is a matrix all of whose entries are zero except for either the $(k-1,k)$–component or the $(k+1,k)$–component.

(4) and (b) if $k=2n$ and $m=2n-1$. In this case, $\varphi(E_{2r-1,2r})$ is a matrix all of whose entries are zero except for the $(2n-1,2n)$–component. So we can get the result of the theorem.

With the same proof as Theorem 2.16, we can get the following theorem.

Theorem 2.17. *Let $\varphi : AlgL_{2n+1} \to AlgL_{2n+1}$ be a surjective isometry such that $\varphi(I)=I$ and $\varphi(E_{ii})=E_{kk}$ and $\varphi(E_{jj})=E_{mm}$ for all $i,j=1,2,\cdots,2n+1$. If $|i-j|=1$, then $|k-m|=1$.*

From Theorems 2.16 and 2.17 we can get the following corollarys.

Corollary 2.18. *Let $\varphi : AlgL_{2n} \to AlgL_{2n}$ (or $AlgL_{2n+1} \to AlgL_{2n+1}$) be a surjective isometry such that $\varphi(I)=I$. Then $\varphi(E_{2r-1,2r})$ and $\varphi(E_{2r+1,2r})$ have the form*

$$\begin{bmatrix} 0 & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 0 & * & & \\ & & & & 0 & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 0 & & & \\ & & & * & 0 & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & 0 \end{bmatrix},$$

where all non–starred entries are zero.

Corollary 2.19. *Let $\varphi : AlgL_{2n} \to AlgL_{2n}$ (or $AlgL_{2n+1} \to AlgL_{2n+1}$) be a surjective isometry such that $\varphi(I)=I$. Then either $\varphi(E_{11})=E_{11}$ or $\varphi(E_{11})=E_{2n,2n}$. (resp. either $\varphi(E_{11})=E_{11}$ or $\varphi(E_{11})=E_{2n+1,2n+1}$).*

Corollary 2.20. *Let $\varphi : AlgL_{2n} \to AlgL_{2n}$ (or $AlgL_{2n+1} \to AlgL_{2n+1}$) be a surjective isometry and $\varphi(E_{ii})=E_{ii}$ for each $i=1,2,\cdots,2n$ (resp. $i=1,2,\cdots,2n+1$). Then there exists a complex number α_{ij} such that $\varphi(E_{ij})=\alpha_{ij}E_{ij}$ for each E_{ij} in $AlgL_{2n}$ (resp. $AlgL_{2n+1}$).*

Define $J : \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ by $J((x_1,x_2,\cdots,x_{2n})^t) = (\bar{x}_{2n},\bar{x}_{2n-1},\cdots,\bar{x}_1)^t$ for every $(x_1,x_2,\cdots,x_{2n})^t$ in $\mathbf{C}^{2n}$. Then J is a conjugation ; that is,

(1) J is a bijection.
(2) $J(x+y) = Jx + Jy$ for all x, y in $\mathbf{C}^{2n}$.
(3) $J(\alpha x) = \bar{\alpha} Jx$ for every α in $\mathbf{C}$ and every x in $\mathbf{C}^{2n}$.
(4) $J^2 = I$.
(5) $\langle Jx, y\rangle = \langle Jy, x\rangle$ for x, y in $\mathbf{C}^{2n}$
(6) $\langle Jx, Jy\rangle = \langle y, x\rangle$.

Define $\varphi_1 : AlgL_{2n} \to AlgL_{2n}$ by $\varphi_1(A) = JA^*J$. Then φ_1 is a surjective isometry and $\varphi_1(I) = I$. From these facts, we get the following Lemma.

Lemma 2.21. *Let $\varphi : AlgL_{2n} \to AlgL_{2n}$ be a surjective isometry such that $\varphi(I) = I$ and $\varphi(E_{11}) = E_{2n,2n}$. Then $\tilde{\varphi} = \varphi_1 \circ \varphi : AlgL_{2n} \to AlgL_{2n}$ is a surjective isometry such that $\tilde{\varphi}(E_{11}) = E_{11}$ and $\tilde{\varphi}(I) = I$.*

Proof. Let $\tilde{\varphi} = \varphi_1 \circ \varphi$ for the above φ_1. Then $\tilde{\varphi}$ is a surjective isometry and $\varphi_1 \circ \varphi(I) = I$ and $\varphi_1 \circ \varphi(E_{11}) = \varphi_1(E_{2n,2n}) = JE_{2n,2n}J = E_{11}$.

Theorem 2.22. *Let $\varphi : AlgL_{2n} \to AlgL_{2n}$ be a surjective isometry such that $\varphi(I) = I$ and $\varphi(E_{11}) = E_{11}$. Then there exists a unitary operator V such that $\varphi(A) = VAV^*$ for all A in $AlgL_{2n}$.*

Proof. From Theorem 2.16 $\varphi(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots, 2n$. From Corollary 2.20 $\varphi(E_{ij}) = \alpha_{ij}E_{ij}$ for all E_{ij} in $AlgL_{2n}$ and some complex number α_{ij}. Since φ is an isometry, $\|\varphi(E_{ij})\| = \|\alpha_{ij}E_{ij}\| = \|E_{ij}\|$ and so $|\alpha_{ij}| = 1$ for all i, j such that E_{ij} in $AlgL_{2n}$. Let $A = (a_{ij})$ be in $AlgL_{2n}$ and let V be a $2n$ by $2n$ diagonal matrix with $e^{i\theta_k}$ the (k,k)–component for all $k(k = 1, 2, \cdots, 2n)$. Then VAV^* is the $2n$ by $2n$ matrix with a_{ii} the (i,i)–component for all $i(i = 1, 2, \cdots, 2n)$, $a_{2j-1,2j}e^{i(\theta_{2j}-1-\theta_{2j})}$ the $(2j-1, 2j)$–component for all $j(j = 1, 2, \cdots, n)$, $a_{2k+1,2k}e^{i(\theta_{2k+1}-\theta_{2k})}$ the $(2k+1, 2k)$–component for all $k(k = 1, 2, \cdots, n-1)$ and 0 the other components. So the theorem will be proved if we can determine $e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_{2n}}$ satisfying the following relations;

$$
\begin{aligned}
e^{i(\theta_1-\theta_2)} &= \alpha_{12} \\
e^{i(\theta_3-\theta_2)} &= \alpha_{32} \\
e^{i(\theta_3-\theta_4)} &= \alpha_{34} \\
&\vdots \\
e^{i(\theta_{2n-1}-\theta_{2n-2})} &= \alpha_{2n-1,2n-2} \\
e^{i(2\theta_{2n-1}-\theta_{2n})} &= \alpha_{2n-1,2n}
\end{aligned}
$$

The equation can be solved recursively (θ_1 may be set equal to 0).

From Lemma 2.21 if $\varphi : AlgL_{2n} \to AlgL_{2n}$ is a surjective isometry such that $\varphi(I) = I$ and $\varphi(E_{11}) = E_{2n,2n}$, then there exists a unitary operator V such that $\varphi_1 \circ \varphi(A) = VAV^*$ for all A in $AlgL_{2n}$, where $\varphi_1 : AlgL_{2n} \to AlgL_{2n}$ is a surjective isometry defined by $\varphi_1(A) = JA^*J$. Hence $\varphi_1 \circ \varphi(A) = VAV^* = J(\varphi(A))^*J$, and so $JVAV^*J = (\varphi(A))^*$. Since $(JAJ)^* = JA^*J$ for all A in $AlgL_{2n}$, we have $\varphi(A) = JVA^*V^*J$.

Lemma 2.23. *Let $\varphi : AlgL_{2n+1} \to AlgL_{2n+1}$ be a surjective isometry such that $\varphi(I) = I$ and $\varphi(E_{11}) = E_{2n+1,2n+1}$. Then there exists a surjective isometry $\varphi_2 : AlgL_{2n+1} \to AlgL_{2n+1}$ such that $\varphi_2(I) = I$ and $\varphi_2 \circ \varphi(E_{11}) = E_{11}$.*

Proof. Let U be a $2n+1$ by $2n+1$ matrix whose $(k, 2n-k+2)$–component is 1 for $k = 1, 2, \cdots, 2n+1$, and all other entries are zero. Define $\varphi_2 : AlgL_{2n+1} \to AlgL_{2n+1}$ by $\varphi_2(A) = U^*AU$. Then φ_2 is a surjective isometry, $\varphi_2(I) = I$ and $\varphi_2 \circ \varphi(E_{11}) = \varphi_2(E_{2n+1,2n+1}) = E_{11}$.

With the same proof as Theorem 2.22, we can get the following theorem.

Theorem 2.24. *Let $\varphi : AlgL_{2n+1} \to AlgL_{2n+1}$ be a surjective isometry such that $\varphi(I) = I$ and $\varphi(E_{11}) = E_{11}$. Then there exists a unitary operator V such that $\varphi(A) = VAV^*$ for all A in $AlgL_{2n+1}$.*

From Theorems 2.23 and 2.24 we can get the following Theorem.

Theorem 2.25. *Let $\varphi : AlgL_{2n+1} \to AlgL_{2n+1}$ be a surjective isometry such that $\varphi(I) = I$ and $\varphi(E_{11}) = E_{2n+1,2n+1}$. Then there is a unitary operator W such that $\varphi(A) = WAW^*$ for all A in $AlgL_{2n+1}$.*

Now by stating a Jo's result we will close this paper. Let L_∞ be the lattice generated by $\{[e_{2i-1}], [e_{2i-1}, e_{2i}, e_{2i+1}]: i = 1, 2, \cdots\}$ and let $\mathcal{A}_\infty$ be the algebra consisting of all bounded operators acting on separable infinite dimensional Hilbert space of the form

$$\begin{bmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \ddots \\ & & & & & \ddots \end{bmatrix},$$

where all non–starred entries are zero, with an orthonormal basis $\{e_1, e_2, \cdots\}$. Then $\mathcal{A}_\infty$ is a reflexive algebra and $AlgL_\infty = \mathcal{A}_\infty$.

Theorem 2.26 [8]. *Let $\varphi : AlgL_\infty \to AlgL_\infty$ be a surjective isometry such that $\varphi(I) = I$. Then there exists a unitary operator V such that $\varphi(A) = VAV^*$ for all A in $AlgL_\infty$.*

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