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HADAMARD PRODUCTS OF BLOCH FUNCTIONS AND ITS DUAL SPACES

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1. Introduction

A function f analytic in the unit disk is said to be of class H^p $(0 < p < \infty)$ if

$$M_p(r,f) = (\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta)^{\frac{1}{p}}, \quad 0 < p < \infty.$$

remains bounded as $r \to 1$.
The class G^p is defined by $f \in G^p$ if and only if $\int_0^1 M_1(r,f')^p dr < \infty$. We shall denote by $\mathcal{B}$ the space consisting of all analytic functions f such that $\sup_{0<r<1}(1-r)|f'(z)| < \infty (r = |z|)$. These functions are called the *Bloch functions*. It is well known that G^1 is the dual space of Bloch functions and $G^1 \subset H^1$.

The Hadamard product of two power series $f(z) = \sum_0^\infty a_n z^n$ and $g(z) = \sum_0^\infty b_n z^n$ is defined as the power series

$$(f * g)(z) = \sum_0^\infty a_n b_n z^n.$$

Let A, B and C be spaces of analytic functions. A complex sequence $\{\lambda_n\}$ is called a multiplier of A into B if $\sum_0^\infty a_n z^n \in A$ implies $\sum_0^\infty \lambda_n a_n z^n \in B$. We denote by (A, B) the space of *multipliers* from A to B. A space of analytic functions can be regarded as a sequence space by identifying each function with its sequence of Taylor coefficients. Then we are easy to see that $C \subset (A, B)$ if and only if $C * A \subset B$.

In this paper, we have the following results.

(1) $\mathcal{B} * \mathcal{B} \subset \mathcal{B}$

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(2) $(H^1, \ell(q,2)) = \ell(q,\infty)$ if $q \geq 1$.
(3) $G^p * G^1 \subset G^p$.
(4) $H^2 * H^1 \subset G^1$.

2. Definitions and notations

Definition 2.1. For $1 \leq \alpha, \beta \leq \infty$ we denote by $\ell(\alpha,\beta)$ the set of those sequences $\{a_k\}(k \geq 1)$ for which

$$\{(\sum_{I_n} |a_k|^\alpha)^{\frac{1}{\alpha}}\}_{n=0}^{\infty} \in \ell^\beta \quad (\alpha < \infty)$$

and

$$\{\sup_{k \in I_n} |a_k|\}_{n=0}^{\infty} \in \ell^\beta \quad (\alpha = \infty)$$

where $I_n = \{k : 2^n \leq k < 2^{n+1}\}$.

See C.N. Kellogg [5] for further information on them. We remark that $\ell(p,p) = \ell^p$.

Definition 2.2. A sequence space A is said to be *solid* if, whenever it contains $\{a_n\}$ it also contains $\{b_n\}$ with $|b_n| \leq |a_n|$. We denote the largest solid subspace of A by $s(A)$.

3. Results

Theorem 3.1. *For $q \geq 1$, $(H^1, \ell(q,2)) = \ell(q,\infty)$.*

Proof. By the result of Paley on the gap series of H^1 functions, we have $H^1 \subset \ell(\infty,2)$. Thus

$$(H^1, \ell(q,2)) \supset (\ell(\infty,2), \ell(q,2)) = \ell(q,\infty).$$

The Köthe dual, denote by $(A)^K$, is defined to be (A, ℓ^1), the multipliers from A to ℓ^1. From Theorem 3.12 ([6]) we have $(G^1)^{KK} = \ell(\infty,1)$. Since $G^1 \subset H^1$,

$$\begin{aligned}(H^1, \ell(q,2)) \subset (G^1, \ell(q,2)) &= ((G^1)^{KK}, \ell(q,2)) \\ &= (\ell(\infty,1), \ell(q,2)) \\ &= \ell(q,\infty),\end{aligned}$$

where we used Lemma 4 of [1].

Corollary 3.2. $\mathcal{B} * H^1 \subset H^2$.

Proof. From Theorem 3.1 we have $(H^1, \ell^2) = \ell(2,\infty)$. Since $\mathcal{B} \subset \ell(2,\infty)$ (see [1]), $\mathcal{B} \subset (H^1, H^2)$. Thus $\mathcal{B} * H^1 \subset H^2$.

Remark. The multipliers $(\ell(\alpha,\beta), \ell(\alpha',\beta')$ are easily determined, this has been done by Kellogg [5].

Theorem 3.3. $\mathcal{B} * \mathcal{B} \subset \mathcal{B}$.

Proof. In [1], $\ell(1,\infty) \subset B \subset \ell(2,\infty)$. Hence $(\ell(2,\infty), \mathcal{B}) \subset (\mathcal{B}, \mathcal{B})$. From Lemma 3 ([1]) we have

$$\begin{aligned}(\ell(2,\infty), \mathcal{B}) &= (\ell(2,\infty), s(\mathcal{B})) \\ &= (\ell(2,\infty), \ell(1,\infty)) \\ &= \ell(2,\infty).\end{aligned}$$

Since $B \subset \ell(2,\infty), \mathcal{B} \subset (\mathcal{B}, \mathcal{B})$. Therefore the space $\mathcal{B}$ is closed under Hadamard product.

Theorem 3.4. $G^p * G^1 \subset G^p$ $(0 < p < \infty)$.

Proof. Let $f(z) = \sum_0^\infty a_n z^n$ be in G^p, let $g(z) = \sum_0^\infty b_n z^n$ be in G^1, and let $h(z) = (f * g)(z) = \sum_0^\infty a_n b_n z^n$. Then

$$h(\rho_z) = (2\pi)^{-1} \int_0^{2\pi} f(\rho e^{it}) g(z e^{-it}) dt. \quad 0 < \rho < 1.$$

Differentiation with respect to z gives

$$\rho h'(\rho z) = (2\pi)^{-1} \int_0^{2\pi} f(\rho e^{it}) g'(z e^{-it}) e^{-it} dt.$$

Hence $\rho M_1(r\rho, h') \le M_1(r, g') M_1(\rho, f)$. Since $g \in G^1, M_1(r, g') = \mathcal{O}(1-r)^{-1}$. Taking $r = \rho$,

$$M_1(r^2, h')^p \le const.(1-r)^{-p} M_1(r, f)^p.$$

Since $f \in G^P, \int_0^1 M_1(r, f')^p dr < \infty$. By the theorem of Hardy and Little–wood on fractional integrals ([4]) we know that $\int_0^1 (1-r)^{-p} M_1(r, f)^p dr < \infty$ if and only if $\int_0^1 M_1(r, f')^p dr < \infty$. Hence $\int_0^1 M_1(r, h')^p dr < \infty$. Thus $h \in G^p$.

Remark. G^1 is the dual space of Bloch functions. Hence the dual space of Bloch functions is closed under the Hadamard product.

Theorem 3.5. $\ell^2 \subset (H^1, G^1) \subset \ell(\infty, 2)$

Proof. By Lemma 6 and Lemma 9 of [1], $s(G^1) = \ell(2,1)$. From Lemma 3 ([1]) and Lemma 4.5 ([6]) we have

$$\begin{aligned}(H^1, G^1) \subset (s(H^1), G^1) &= (\ell^2, G^1) \\ &= (\ell^2, s(G^1)) = (\ell^2, \ell(2,1)) \\ &\doteq \ell(\infty, 2).\end{aligned}$$

Since $H^1 \subset \ell(\infty, 2)$,

$$\begin{aligned}(H^1, G^1) &\supset (\ell(\infty,2), G^1) = (\ell(\infty,2), s(G^1)) \\ &= (\ell(\infty,2), \ell(2,1)) \\ &= \ell(2,2) = \ell^2.\end{aligned}$$

Corollary 3.6. $H^2 * H^1 \subset G^1$

Remark. In [1], $\ell(1,\infty) = s(\mathcal{B}) \subset \mathcal{B} \subset \ell(2,\infty)$. By the similar method of Theorem 3.5, $(\mathcal{B}, G^1) = \ell(\infty, 1)$. Since $\ell(2,1) \subset G^1 \subset \ell(\infty,1)$ (see [1] p.263), $G^1 \subset (\mathcal{B}, G^1)$. Hence $\mathcal{B} * G^1 \subset G^1$.

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