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On the Subgroups of Fundamental Group in the Fixed Point Theory

by

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1. Introduction

Let X be a topological space with x_o as a base point. A homotopy $h_t : X \longrightarrow X$ $(t \in I)$ is called a *cyclic homotopy based at* 1_X (identity map), if $h_o = h_1 = 1_X$. Let $f: X \longrightarrow X$ be a given continuous map such that $f(x_o) = x_o$. A homotopy $h_t : X \longrightarrow X$ is called a *cyclic homotopy based at* f if $h_o = h_1 = f$. If h_t is a cyclic homotopy (based at 1_X or based at f), the path σ: $I \longrightarrow X$ given by $\sigma(t) = h_t(x_o)$ is a loop which will be called the *trace of* h_t.

The set of homotopy classes of those loops which are the trace of some cyclic homotopy based at 1_X form a subgroup of the fundamental group $\Pi_1(X, x_o)$ which we denote by $G(X, x_o)$ ([4], p. 842).

The set of homotopy classes of those loops which are the trace of some cyclic homotopy based at f form a subgroup of $\Pi_1(X, x_o)$ which we denote by $T(f, x_o)$ ([8], p. 31).

The set of elements of $\Pi_1(X, f(x_o)) = \Pi_1(X, x_o)$ which operate trivially on $f_* \Pi_n(X, x_o)$ for all $n \geq 1$ form a subgroup of $\Pi_1(X, x_o)$ which is denoted by $P(X, f, x_o)$ ([3], p. 9).

A lot of good properties of the above subgroups are investigated and are used in dealing with the numbers of fixed points of f: $X \longrightarrow X$.

In the present paper, we deal with the properties of $P(X, f, x_o)$, the computations

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of $P(X, f, x_o)$, and the relations among the above subgroups.

2. Preliminaries

We will introduce some terminologies and notations, and will summarize the results essential to our main results.

A topological space X is a polyhedron if there exists a simplicial complex K such that $|K|$ is homeomorphic to X.

Definition 1. The set of elements of $\Pi_1(X, x_o)$ which operate trivially on $\Pi_n(X, x_o)$ for $n \geq 1$ ([6], p. 126) form a subgroup of $\Pi_1(X, x_o)$ which will be denoted by $P(X, x_o)$ ([4], p. 843).

By the above definitions, we have $P(X, x_o) \subseteq P(X, f, x_o)$.

Definition 2. For a given integer $n > 0$, a path-connected space X is said to be *n-simple* if there exists a point $x_o \in X$ such that $\Pi_1(X, x_o)$ operates trivially on $\Pi_n(X, x_o)$.

The above two definitions lead to the following:

A space X is n-simple for all $n > 1$ if and only if $P(X, x_o) = \Pi_1(X, x_o)$.

Sometimes we will write $\Pi_1(X)$ rather than $\Pi_1(X, x_o)$ for brevity, if the base point is unimportant.

In the sequel we will generally assume X to be a connected compact polyhedron and the continuous map $f: X \longrightarrow X$ to be $f(x_o) = x_o$. We may assume that the maps with which we deal fix the base point x_o, since we consider the space X in the view point of fixed point theory and we have complete freedom of choice of the base point for the subgroups of $\Pi_1(X, x_o)$, ([1], p. 17).

3. Statement of the main results

Since any element of $T(f, x_o)$ operates trivially on $f_* \Pi_n(X, x_o)$ for all $n \geq 1$ ([1], p. 44), we have the following:

$$T(f,x_o)\subseteq P(X,f,x_o). \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (A)$$

The subgroup of $\Pi_1(X,x_o)$ which operates trivially on $f_*\Pi_1(X,x_o)$ is precisely the centralizer of $f_*\Pi_1(X,x_o)$ in $\Pi_1(X,x_o)$, denoted by $Z(f_*\Pi_1(X),\ \Pi_1(X))$.

Thus we have the following:

$$P(X,f,x_o)\subseteq Z(f_*\Pi_1(X),\ \Pi_1(X)).$$

By the above fact and (A), we have

$$T(f,x_o)\subseteq P(X,f,x_o)\subseteq Z(f_*\Pi_1(X),\ \Pi_1(X))\subseteq \Pi_1(X,x_o). \cdots\cdots\cdots (B)$$

If f_* is an epimorphism for all $n\geq 1$, that is, $f_*\Pi_n(X,x_o)=\Pi_n(X,x_o)$, then by the definitions we have

$$P(X,x_o)=P(X,f,x_o).$$

Theorem 1. If X is a connected aspherical polyhedron(in the sense of $\Pi_n(X,x_o)=0$ for $n>1$) and $f\colon X\longrightarrow X$ is a continuous map, then we have

$$T(f,x_o)=P(X,f,x_o)=Z(f_*\Pi_1(X),\ \Pi_1(X)).$$

Proof. In the case that X is aspherical, $\Pi_n(X,x_o)$ is the trivial group for $n>1$ (denoted by 0). So we may consider the case of $n=1$ only, and we have $P(X,f,x_o)$ $=Z(f_*\Pi_1(X),\ \Pi_1(X))$. On the other hand, we have $Z(f_*\Pi_1(X),\ \Pi_1(X))\subseteq T(f,x_o)$ ([2], p. 102).

Thus we have $P(X,f,x_o)=Z(f_*\Pi_1(X),\ \Pi_1(X))\subseteq T(f,x_o)\subseteq P(X,f,x_o)$. ///

Moreover, if X is a connected aspherical polyhedron and f_* is an epimorphism, then we have the following:

$T(f,x_o)=P(X,x_o)=P(X,f,x_o)=Z(\Pi_1(X))=Z(f_*\Pi_1(X),\ \Pi_1(X))$, where $Z(\Pi_1(X))$ means the center of $\Pi_1(X,x_o)$.

By Theorem 1, we can see the following result: If X is connected aspherical, then $P(X,f,x_o)=\Pi_1(X,x_o)$ implies $T(f,x_o)=\Pi_1(X,x_o)$, that is, $f\colon X\longrightarrow X$ satisfies the Jiang condition, and every nice results of Jiang follow.

The subgroup of $\Pi_1(X,x_o)$ which operates trivially on $\Pi_1(X,x_o)$ itself is precisely the center of $\pi_1(X, x_o)$. Thus we have

$$P(X, x_o) \subseteq Z(\pi_1(X)).$$

We have $G(X, x_o) \subseteq P(X, x_o)$ ([4], p.843). So we have

$$G(X, x_o) \subseteq P(X, x_o) \subseteq Z(\pi_1(X)) \subseteq Z(f_* \pi_1(X), \pi_1(X)) \subseteq \pi_1(X, x_o) \cdots \text{(C)}$$

On the other hand, we have

$$G(X, x_o) \subseteq T(f, x_o) \quad ([2],\ \text{p.} 101). \cdots\cdots\cdots\cdots \text{(D)}$$

An H-space consists of a pointed topological space (X, e) together with a continuous multiplication and an element e such that right and left multiplication are both homotopic to the identity of X.

Proposition 2. If X is an H-space, then

$$G(X, x_o) = T(f, x_o) = P(X, x_o) = P(X, f, x_o) = Z(\pi_1(X)) = \pi_1(X, x_o).$$

Proof. Since $G(X, x_o) = \pi_1(X, x_o)$ ([4], p.844), the result follows from (B), (C), (D), and the fact of $P(X, x_o) \subseteq P(X, f, x_o)$. ///

We compute $P(X, f, x_o)$ for some special spaces.

For the circle S^1 and the 1-dimensional torus T, we have

$$P(S^1, f, x_o) \cong Z \text{ and } P(T, f, x_o) \cong Z \times Z,$$

because they are both H-spaces. For the sphere S^2, we have $P(S^2, f, x_o) = 0$ because of the $\pi_1(S^2) = 0$.

Theorem 3. Let X be connected aspherical and let $f: X \longrightarrow X$ be any continuous map. Then

$$P(X, f, x_o) = \pi_1(X, x_o) \text{ if and only if } f_* \pi_1(X) \subseteq Z(\pi_1(X)).$$

Proof. Assume $P(X, f, x_o) = \pi_1(X, x_o)$. Then we have $Z(f_* \pi_1(X), \pi_1(X)) = \pi_1(X, x_o)$. This means that every element $\alpha \in \pi_1(X, x_o)$ commutes with every element $(f\sigma) \in f_* \pi_1(X)$, and thus we have $f_* \pi_1(X) \subseteq Z(\pi_1(X))$. Conversely assume $f_* \pi_1$

$(X) \subseteq Z(\pi_1(X))$. Then every element commutes with any element of $f_* \pi_1(X)$.

So $\pi_1(X, x_0)$ operates trivially on $f_* \pi_1(X)$. Since X is aspherical, we may consider the case of $n=1$ only, and we have $P(X, f, x_0) = \pi_1(X, x_0)$. ///

A graph is a 1-dimensional simplicial complex.

Lemma 4. Let X be a connected compact graph. Then for any continuous map $f: X \longrightarrow X$, we have

$$T(f, x_0) = P(X, f, x_0) = Z(f_* \pi_1(X), \pi_1(X)).$$

Proof. Since X is 1-dimensional, we have $\pi_n(X, x_0) = 0$ for $n > 1$, and thus X is aspherical. The conclusion follows from Theorem 1.

Theorem 5. Let X be a connected compact graph that is not of the same homotopy type as S^1. Suppose f_* is an epimorphism. Then $T(f, x_0) = P(X, f, x_0) = 0$.

Proof. The fundamental group of a graph is a free group ([5], 6.3.11) and the only graph X such that $Z(\pi_1(X))$ is nontrivial has the same homotopy type as S^1. But, X is not of the same homotopy type as S^1. So we have $Z(\pi_1(X)) = 0$.

Consequently, by Lemma 4 we have $T(f, x_0) = P(X, f, x_0) = 0$. ///

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